

A CLASSIFICATION OF COVERINGS YIELDING HEUN-TO-HYPERGEOMETRIC REDUCTIONS

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ABSTRACT. Pull-back transformations between Heun and Gauss hypergeometric equations give useful expressions of Heun functions in terms of better understood hypergeometric functions. This article classifies, up to Möbius automorphisms, the coverings $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ that yield pull-back transformations from hypergeometric to Heun equations with at least one free parameter (excluding the cases when the involved hypergeometric equation has cyclic or dihedral monodromy). In all, 61 parametric hypergeometric-to-Heun transformations are found, of maximal degree 12. Among them, 28 pull-backs are compositions of smaller degree transformations between hypergeometric and Heun functions. The 61 transformations are realized by 48 different Belyi coverings (though 2 coverings should be counted twice as their moduli field is quadratic). The same Belyi coverings appear in several other contexts. For example, 38 of the coverings appear in Herfutner's list of elliptic surfaces over \mathbb{P}^1 with four singular fibers, as their j -invariants. In passing, we demonstrate an elegant way to show that there are no coverings $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ with some branching patterns.

1. CONTEXT AND OVERVIEW

The Gauss hypergeometric equation

$$(1.1) \quad \frac{d^2 y(z)}{dz^2} + \left(\frac{C}{z} + \frac{A+B-C+1}{z-1} \right) \frac{dy(z)}{dz} + \frac{AB}{z(z-1)} y(z) = 0$$

and the Heun equation

$$(1.2) \quad \frac{d^2 Y(x)}{dx^2} + \left(\frac{c}{x} + \frac{d}{x-1} + \frac{a+b-c-d+1}{x-t} \right) \frac{dY(x)}{dx} + \frac{abx-q}{x(x-1)(x-t)} Y(x) = 0$$

are canonical second-order Fuchsian differential equations on the Riemann sphere \mathbb{P}^1 , with 3 and 4 regular singularities, respectively. Transformations among these equations give identities between their standard hypergeometric and Heun solutions. For example, there is a single covering $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree 2 (up to Möbius transformations). It induces the classical quadratic transformations of hypergeometric functions, such as

$$(1.3) \quad {}_2F_1 \left(\begin{matrix} 2A, 2B \\ A+B+\frac{1}{2} \end{matrix} \middle| x \right) = {}_2F_1 \left(\begin{matrix} A, B \\ A+B+\frac{1}{2} \end{matrix} \middle| 4x(1-x) \right).$$

Moreover, the same covering induces the well-known Heun-to-Heun quadratic transformation [17, Thm. 4.1], and an identification of the general ${}_2F_1(A, B; C | 4x(1-x))$ function with a standard local solution of Heun's equation with the parameters $(t, q, a, b, c, d) = (\frac{1}{2}, 2AB, 2A, 2B, C, C)$. These transformations are *parametric*, since they have at least one free parameter such as A, B .

The aim of this paper is classification of all parametric pull-back transformations between hypergeometric and Heun functions. The considered pull-back transformations are of the form

$$(1.4) \quad z \mapsto \varphi(x), \quad y(z) \mapsto Y(x) = \theta(x) y(\varphi(x)),$$

where $\varphi(x)$ is a rational function and $\theta(x)$ is a radical function, i.e., a product of powers of rational functions. Geometrically, transformation (1.4) *lifts* or *pulls back* a Fuchsian equation on the curve \mathbb{P}_z^1 to one on the curve \mathbb{P}_x^1 , along the covering $\varphi: \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$. The *gauge prefactor* $\theta(x)$ is usually chosen such that the pulled-back equation has fewer singularities and canonical values of some local exponents.

Pull-back transformations between Gauss hypergeometric equations were recently classified by Vidunas [29]. Next to the classical quadratic, cubic and Goursat [8] transformations, a few sets of unpredicted transformations were found, including parametric transformations from hypergeometric equations with cyclic or dihedral monodromy. Moreover, the hypergeometric-to-Heun transformations without the prefactor $\theta(x)$ have been classified by Maier [16]. In both classifications, the heart of the problem is determining the covering maps $\varphi(x)$ that can appear. They are typically *Belyi maps*, in the sense that (apart from dull exceptions of Proposition 2.3 here) they have at most 3 critical values on the Riemann sphere \mathbb{P}_z^1 . In fact, the critical values of those $\varphi(x)$ are typically the singular points $z = 0$, $z = 1$, $z = \infty$ of the hypergeometric equation, and the branching points include the singularities $x = 0$, $x = 1$, $x = \infty$ (and $x = t$) of the pulled-back hypergeometric (or Heun) equation. The approaches of [16, 29] include:

- (i) determining the *branching patterns* that φ can have;
- (ii) determining which of those patterns can be *realized* by a rational function $\varphi(x)$;
- (iii) normalizing the points $x = 0$, $x = 1$, $x = \infty$ of $\varphi(x)$, and deriving identities between hypergeometric and Heun functions by identifying corresponding local solutions of thereby related differential equations.

This article follows this strategy and the techniques of [29] to generate a complete list of coverings φ that can appear in parametric Heun-to-hypergeometric reductions. We find 61 different transformations (excluding infinite families of pull-backs from hypergeometric equations with cyclic or dihedral monodromy [32]), realized by 48 different Belyi coverings. An explicit formula for each covering is given in Table 4. The Belyi maps are not normalized for Step (iii). The induced identities between hypergeometric and Heun functions are thoroughly examined in the parallel article [30]. Here we are not concerned with the technical issues of determining the prefactor $\theta(x)$, identifying local solutions, symmetries of the hypergeometric and Heun equations, nor even introducing Heun functions.

By the Grothendieck correspondence [22] any Belyi map $\varphi: \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$ corresponds bijectively to a *dessin d'enfant* on \mathbb{P}_x^1 , up to Möbius isomorphisms of both Riemann spheres \mathbb{P}_z^1 , \mathbb{P}_x^1 . Generally, the dessins are defined combinatorially as certain bicolored graphs. For our purposes, the *dessins d'enfant* of a Belyi map $\varphi(x)$ is the graph on \mathbb{P}_x^1 obtained as the pre-image of the line segment $[0, 1]$ on \mathbb{P}_z^1 , up to isotopy. The vertices above $z = 0$ are colored black, and the vertices above $z = 1$ are colored white. The order of each vertex is equal to the branching order at the corresponding x -point. Figure 1 depicts the dessins for all 48 encountered Belyi coverings. Most of the white points have order 2, and then they are not depicted. Black points of

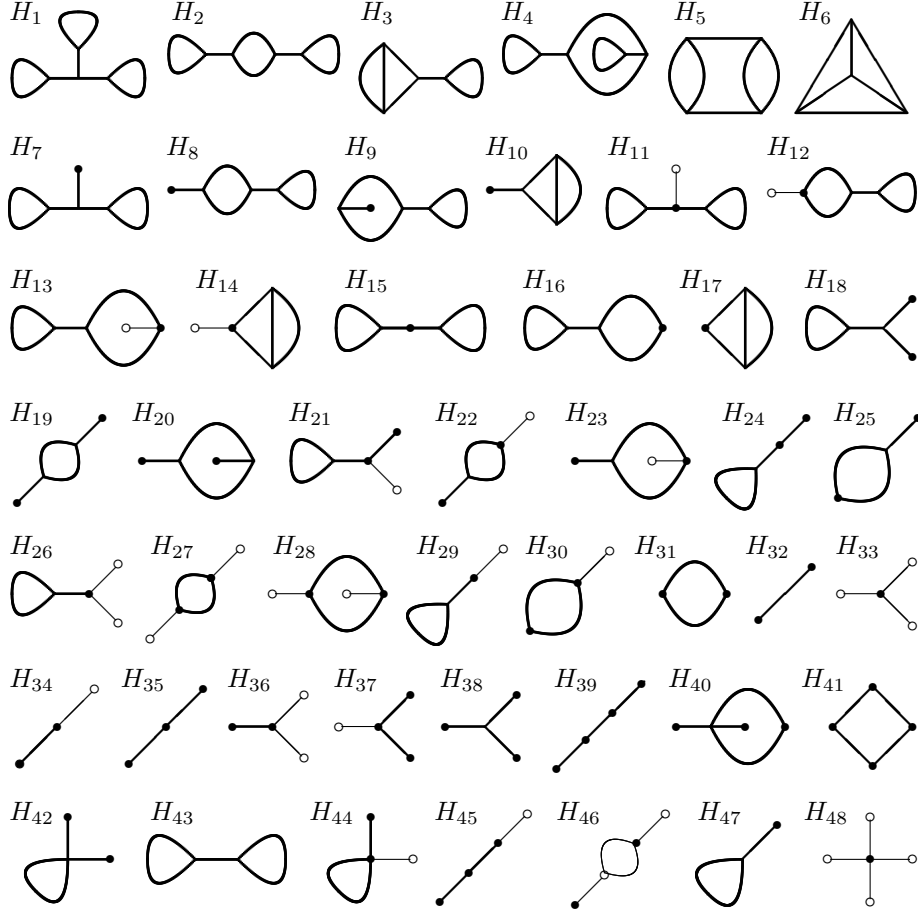


FIGURE 1. *Dessins d'enfant* of the Belyi coverings for parametric Heun-to-hypergeometric reductions

order 3 or 4 are not depicted either, unless they are connected to a white point of order 1. A thin edge connects a pair of displayed black and white vertices. A thick edge connects two black points (either displayed or clearly branching) with an implicit white point somewhere in the middle. Each *cell* (i.e., a two-dimensional connected component of the complement on \mathbb{P}_x^1 , possibly the outer one) represents a point above $z = \infty$. The branching order of each cell is determined by counting the number of black points met while tracing a loop along its boundary.

It is instructive to follow the branching orders and incidences on the dessins while following our classification of possible coverings in Tables 1–3. In principle, the pull-back Belyi coverings can be classified by generating and counting the dessins satisfying the suitable branching patterns. However, it is difficult to ensure completeness of a large list of dessins. We first computed the Belyi coverings explicitly, then easily generated the required dessins by combinatorial consideration. For each possible branching pattern, there is at most one Belyi covering except for the

coverings H_{21} and H_{44} . Therefore completeness and identification of the dessins is quickly established. The coverings H_{21} , H_{44} are defined over $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(i)$, respectively. All other coverings are defined over \mathbb{Q} and \mathbb{R} , hence their dessins have a reflection symmetry. The dessins for H_{21} , H_{44} should actually be counted twice, as the complex conjugation gives non-isotopic dessins. The proper count of dessins and Belyi coverings is therefore 50, not 48.

Many of the encountered Belyi coverings occur in other contexts, particularly in the theory of elliptic surfaces and Picard-Fuchs equations. The coverings from H_1 to H_{38} occur in Herfurtners list [10] of elliptic surfaces with four singular fibers, up to Möbius transformations. The order of these coverings follows [10, Table 3], and the numbering is used in [19] where the corresponding pull-backs to Heun equations (specializable to Picard-Fuchs equations for the elliptic surfaces) are observed. The coverings H_1 to H_6 have the maximal degree 12, and produce the Beauville list [2] of the coverings generating *semi-stable* elliptic surfaces with four singular fibers. Their branching orders above $z = 0$ are all 3, and above $z = 1$ they are all 2, as can be seen from the dessins. The branching pattern of H_1 is written by us as follows:

$$(1.5) \quad [2]_6 = [3]_4 = 9+1+1+1.$$

The four singular fibers of the corresponding elliptic surface have the Kodaira types I_9, I_1, I_1, I_1 . This covering is also described as a *Davenport-Stothers triple* [25]: it can be written as F^3/G^2 , where F, G are polynomials of degree 4 and 6 (respectively), such that the polynomial $F^3 - G^2$ has the minimal possible degree 3.

A pull-back transformation defined over \mathbb{R} can be nicely illustrated by subdivisions of the Schwarz quadrangle for the pulled-back Heun equation into Schwarz triangles for the initial hypergeometric equations, following [11, 12]. In the hyperbolic geometry setting, these are *Coxeter decompositions* [7] or *divisible tilings* [5] of a hyperbolic quadrangle into mutually similar hyperbolic triangles. We describe these picturesque illustrations in §4.3 and Figure 2.

This article is structured as follows. Section 2 establishes pivotal lemmas on the behavior of singularities and local exponents of Fuchsian equations under pull-back transformations. Section 3 presents the main results in Tables 1–4, and explains them (and the notation) in a few steps. Of the three mentioned generation steps (i)–(iii), the first step is elaborated in §§3.1, 3.2, while computations for Step (ii) are reviewed in §4.1. Step (iii) is thoroughly considered in the parallel paper [30]. Furthermore, §4 relates our classification to Herfurtners list [10] and Felikson’s list of Coxeter decompositions [7], and §4.4 examines the composite transformations. Section 5 presents an elegant approach to prove non-existence (or uniqueness) of Belyi coverings with some branching patterns, and applies it not only to the obtained list of branching patterns, but also to the Miranda–Persson classification [18] of K3 semi-stable elliptic surfaces with six singular fibers.

2. PULL-BACKS AND LOCAL EXPONENTS

The singular points and the local exponents of Gauss hypergeometric equation (1.1) are usefully encoded in the Riemann P-symbol scheme

$$(2.1) \quad P \left\{ \begin{array}{ccc|c} 0 & 1 & \infty & z \\ 0 & 0 & a & \\ 1-c & c-a-b & b & \end{array} \right\}.$$

The local exponent differences at the 3 singular points are therefore

$$(2.2) \quad 1 - c, \quad c - a - b, \quad a - b.$$

Similarly, the Riemann scheme of the Heun equation (1.2) is

$$(2.3) \quad P \left\{ \begin{array}{cccc|c} 0 & 1 & t & \infty & x \\ 0 & 0 & 0 & a & \\ 1 - c & 1 - d & c + d - a - b & b & \end{array} \right\}.$$

The parameters a, b, c, d determine the local exponents, while the parameter q is *accessory*. In particular, the 4 exponent differences are

$$(2.4) \quad 1 - c, \quad 1 - d, \quad c + d - a - b, \quad a - b.$$

The Heun equation contains many interesting special cases, including the Lamé equation [6]. The Heun equation and its solutions appear in problems of diffusion, wave propagation, heat and mass transfer, magneto-hydrodynamics, particle physics, and the cosmology of the very early universe.

By $E(\alpha, \beta, \gamma)$ we denote a Gauss hypergeometric equation of the form (1.1) with the exponent differences (2.2) equal to α, β, γ in some order. Similarly, by $HE(\alpha, \beta, \gamma, \delta)$ we denote a Heun equation of the form (1.2) with its exponent differences equal to $\alpha, \beta, \gamma, \delta$ in some order. These notations do not assign local exponents to particular singular points, nor they specify the accessory parameter q .

The degree of a pull-back transformation (1.4) between Fuchsian equations is the degree of the rational function $\varphi(x)$. The existence of a pull-back from some $E(\alpha_1, \beta_1, \gamma_1)$ to some $HE(\alpha_2, \beta_2, \gamma_2, \delta_2)$ of degree D will be indicated by

$$(2.5) \quad E(\alpha_1, \beta_1, \gamma_1) \xleftarrow{D} HE(\alpha_2, \beta_2, \gamma_2, \delta_2).$$

Sometimes the pull-back covering or the transformation will be indicated more specifically by a subscript on the degree D . Similarly,

$$E(\alpha_1, \beta_1, \gamma_1) \xleftarrow{D} E(\alpha_2, \beta_2, \gamma_2), \quad HE(\alpha_1, \beta_1, \gamma_1, \delta_1) \xleftarrow{D_H} HE(\alpha_2, \beta_2, \gamma_2, \delta_2)$$

will indicate pull-back transformations between hypergeometric or between Heun equations. For brevity, we refer to these three types of transformations as *Gauss-to-Heun*, *Gauss-to-Gauss* (or just *hypergeometric*) and *Heun-to-Heun* pull-back transformations. In particular, the 3 quadratic transformations mentioned at the beginning of this article actually are:

$$(2.6) \quad E(1/2, \alpha, \beta) \xleftarrow{2} E(\alpha, \alpha, 2\beta),$$

$$(2.7) \quad HE(1/2, 1/2, \alpha, \beta) \xleftarrow{2_H} HE(\alpha, \alpha, \beta, \beta),$$

$$(2.8) \quad E(\alpha, \beta, \gamma) \xleftarrow{2} HE(\alpha, \alpha, 2\beta, 2\gamma).$$

As in the notation $(\alpha_1, \beta_1, \gamma_1) \xleftarrow{D} (\alpha_2, \beta_2, \gamma_2)$ of [29], the arrows follow the direction of the covering $\varphi: \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$. To emphasize: these notations indicate the existence of *some* differential equations with the stated exponent differences that are related by a pull-back transformation, rather than the existence of a pull-back between *any* equations with the specified exponent differences.

Our classification is obtained by considering the behavior of singularities and local exponents of Fuchsian equations under pull-backs. Any transformation of the form (1.4) pulls-back a Fuchsian equation to a Fuchsian equation, usually with more singular points. To pull-back a hypergeometric equation to a Fuchsian equation

with just 4 singular points, special restrictions apply to the covering $\varphi(x)$ and the hypergeometric equation.

The following definitions are taken from [29]. An *irrelevant singular point* of a Fuchsian equation is a non-logarithmic singular point where the local exponent difference is equal to 1. For comparison, an *ordinary* (i.e., non-singular) point is a non-logarithmic point with the local exponents 0 and 1, and an *apparent singularity* is a non-logarithmic singular point with the local exponents 0 and an integer $k > 1$. A *relevant singular point* is one that is not irrelevant. Any irrelevant singular point can be turned into an ordinary point by a pull-back (1.4) which is prefactor-only, i.e., one with $\varphi(x) = x$. Hence, what is of primary importance is how many *relevant* singular points the pulled-back equation has. This number is affected only by the choice of covering $\varphi(x)$, and not by the choice of prefactor $\theta(x)$.

The following two lemmas describe the crucial behavior of singularities and local exponents under pull-backs.

Lemma 2.1. *Let $\varphi: \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$ be a finite covering. Let E_1 denote a Fuchsian equation on \mathbb{P}_z^1 , and let E_2 denote the pull-back on \mathbb{P}_x^1 of E_1 by transformation (1.4). For any $S \in \mathbb{P}_x^1$, let $k := \text{ord}_\varphi(P)$ denote the branching order of φ at S .*

- (a) *The exponents of E_2 at S equal $k\alpha_1 + \gamma$, $k\alpha_2 + \gamma$, where:

 - α_1, α_2 are the exponents of E_1 at $\varphi(S) \in \mathbb{P}_z^1$;
 - γ is the exponent of the radical function $\theta(x)$ at S .*
- (b) *If $\varphi(S)$ is an ordinary point of E_1 , then S will fail to be a relevant singular point for E_2 if and only if $k = 1$ (i.e., the covering φ does not branch at S , i.e., S is not a branching point of φ).*
- (c) *If $\varphi(S)$ is a singular point of E_1 , then S will fail to be a relevant singular point of E_2 if and only if

 - $k > 1$ and the exponent difference at $\varphi(S)$ is equal to $1/k$; or,
 - $k = 1$ and $\varphi(S)$ is irrelevant.*

In either case S will be an irrelevant singular point or an ordinary point.

Proof. The first statement is mentioned in the proof of [29, Lemma 2.4]. The other two statements are parts 2 and 3 of [29, Lemma 2.4]. \square

Lemma 2.2. *Let $\varphi: \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$ be a covering of degree D , and let Δ denote a set of 3 points on \mathbb{P}_z^1 .*

- (a) *If all branching points of φ lie above Δ , i.e., no point of $\mathbb{P}_z^1 \setminus \Delta$ is a critical value of φ , then there are exactly $D + 2$ distinct points on \mathbb{P}_x^1 above Δ . Otherwise, there are more than $D + 2$ distinct points above Δ .*
- (b) *If there are exactly $D + 3$ distinct points above Δ , there is only one branching point that is not above Δ .*

Proof. The first statement is part 1 of [29, Lemma 2.5]. It follows from the Hurwitz formula [9, Corollary IV.2.4], which says that the sum of $\text{ord}_\varphi(P) - 1$ over the branching points $P \in \mathbb{P}_x^1$ must equal $2(D - 1)$. The second statement is a slight extension (utilized in [13]). \square

Suppose one starts with a hypergeometric equation E_1 on \mathbb{P}_z^1 . Let Δ denote the set $\{0, 1, \infty\}$ containing the singularities of E_1 . It follows from the above lemmas that to minimize the number of singular points of a pull-back of E_1 , one should typically allow branching points of φ only above Δ . Otherwise, there would be more than $D + 2$ distinct points above Δ , and generically, each of these $D + 2$

points would be a singular point of the pulled-back equation. By Lemma 2.1(c), further minimization is possible if one or more of the exponent differences of E_1 in Δ are restricted to be of the form $1/k$.

Recall that a covering $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a *Belyi covering* [24] if it is unbranched above the complement of a set of three points, such as $\{0, 1, \infty\}$. By the above consideration, one expects that the pull-back coverings for Gauss-to-Heun transformations will typically be Belyi coverings. The following proposition classifies the rather degenerate situations in which non-Belyi coverings can occur.

Proposition 2.3. *Suppose there is a pull-back transformation (1.4) of a hypergeometric equation E_1 to a Fuchsian equation with at most 4 singular points, and the covering defined by the rational function $\varphi(x)$ is not a Belyi map. Then one of the following statements must hold:*

- (i) *Two of the three exponent differences of E_1 are equal to $1/2$; or*
- (ii) *E_1 has a basis of solutions consisting of algebraic functions of z .*

Proof. Let $D = \deg \varphi$, and $\Delta = \{0, 1, \infty\} \subset \mathbb{P}_z^1$. Since $\varphi: \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$ is not a Belyi map, there is a branching point P_0 that does *not* lie above Δ . By part (a) of Lemma 2.2, there are at least $D + 3$ distinct points above Δ . At most 3 of them can be singularities of the pulled-back equation, because P_0 will be a singularity by Lemma 2.1(b). Therefore there are at least D ordinary points above Δ .

One or more of the 3 exponent differences of E_1 must be of the form $1/k$ for an integer $k \geq 1$, because only then ordinary points occur above Δ by Lemma 2.1(c). Above a point of Δ with the exponent difference $1/k$, there may be at most D/k ordinary points. Let M denote the number of restricted exponent differences of E_1 . There are three possibilities:

- $M = 1$. One must have $k = 1$, and by Lemma 2.1(c), this point is not a relevant singularity for E_1 . Let m denote the number of distinct points above the two (generally) relevant singularities of E_1 . If $m = 2$, the covering is cyclic (i.e., Möbius-equivalent to $\varphi(x) = x^D$). If $m = 3$, there is only one branching point not above the relevant singularities of E_1 , by Lemma 2.2(b) basically. Hence φ is a Belyi covering for $m \leq 3$. If $m > 3$, the pulled-back equation will have more than 4 singularities.
- $M = 2$. The exponent differences will be $1/k, 1/\ell$ with k, ℓ positive integers and $D/k + D/\ell \geq D$. One must have $1/k, 1/\ell = 1/2$, which is case (i).
- $M = 3$. The exponent differences will be $1/k, 1/\ell, 1/m$ with k, ℓ, m positive integers and $D/k + D/\ell + D/m \geq D$, i.e., $1/k + 1/\ell + 1/m \geq 1$. The subcase $1/k + 1/\ell + 1/m = 1$ can be ruled out, since even if the points above Δ are optimally arranged, there will be fewer than D ordinary points above Δ , contradicting Lemma 2.2(a). It is known [6, 20] that in the subcase $1/k + 1/\ell + 1/m > 1$, the equation E_1 has only algebraic solutions. \square

Remark. In case (i), the projective monodromy group of E_1 is generally an infinite dihedral group. As we recall in §5, the possible projective monodromies in case (ii) are: a finite cyclic, a finite dihedral, A_4 (tetrahedral), S_4 (octahedral) or A_5 (icosahedral) groups. If $M = 1$, the monodromy is generally an infinite cyclic group. Gauss-to-Heun transformations with (finite or infinite) cyclic or dihedral monodromy are going to be considered throughly in a separate article [32].

Exponent differences		Deg. D	Branching pattern above singularities	Covering characterization, composition
hyperg.	Heun			
α, β, γ	$\alpha, \alpha, 2\alpha, 2\gamma$	2	$2 = 1+1 = 1+1$	H_{32}, F_1, F_1^* , indecomposable
$1/2, \alpha, \beta$	$\alpha, \alpha, 2\alpha, 4\beta$	4	$[2]_2 = 4 = 2+1+1$	$H_{35}, F_4, F_5^*, 2 \cdot 2$
	$\alpha, 3\alpha, \beta, 3\beta$		$[2]_2 = 3+1 = 3+1$	H_{47}, F_4', F_6^* , indecomposable
	$2\alpha, 2\alpha, \beta, 3\beta$		$[2]_2 = 3+1 = 2+2$	no covering, N_{27}
	$2\alpha, 2\alpha, 2\beta, 2\beta$		$[2]_2 = 2+2 = 2+2$	$H_{31}, F_3, F_4^*, 2 \times 2$
	$1/2, \alpha, 2\alpha, 3\beta$	3	$[2]_1 + 1 = 2+1 = 3$	H_{34}, F_2, F_2^* , indecomposable
$1/3, \alpha, \beta$	$\alpha, 2\alpha, \beta, 2\beta$	3	$[3]_1 = 2+1 = 2+1$	H_{34}, F_2'', F_3^* , indecomposable
	$\alpha, \alpha, \alpha, 3\beta$		$[3]_1 = 3 = 1+1+1$	H_{33} , indecomposable

TABLE 1. Possible branching patterns of hypergeometric-to-Heun transformations with 2 or 3 free parameters.

3. MAIN RESULT: GENERATION AND CLASSIFICATION

Here we present the method and the results of classification of Gauss-to-Heun transformations with at least one free parameter. Following part (c) of Lemma 2.1, we restrict $m \in \{0, 1, 2\}$ local exponent differences of the general hypergeometric equation (1.1) to the reciprocals of integers $k \geq 1$. Thereby we have $M = 3 - m$ free parameters. Basically, the free parameters are the unrestricted exponent differences.

We ignore the cases (considered in [32]) when an exponent difference is restricted to 1 at a non-logarithmic singularity, or when two exponent differences are restricted to $1/2$, as then the hypergeometric equation has cyclic or dihedral monodromy. Apart from this, Tables 1, 2, 3 below give a full list of Gauss-to-Heun pull-back transformations with a free parameter in terms of the exponent differences (in the first two columns), the degree and the branching pattern of the pull-back covering (in the next two columns) among the entries where a covering is indicated by the H -notation in the last column. Table 4 gives a full list of the encountered Belyi maps (up to Möbius transformations and complex conjugation), and the introductory Figure 1 depicts the *dessins d'enfant* of those Belyi maps. The parallel article [30] identifies the pulled-back Heun equations in detail, and gives a representative list of transformation formulas between hypergeometric and Heun functions.

We proceed to explain the results and notation in Tables 1–4. Let ω denote a primitive cubic root of unity, say $\omega = \exp(2\pi i/3)$. In particular, $\omega^2 + \omega + 1 = 0$.

The pull-back transformations from a hypergeometric equation E_1 to a Heun equation E_2 are classified and demonstrated in the following four steps. They parallel the principal steps (i)–(iii) outlined in the introduction, with the only difference that Step (i) is split into two steps.

Step 1 is determination of possible restrictions on the exponent differences of E_1 and the degree of the pull-backs. This step is elaborated in §3.1. The restrictions on the exponent differences determine the *type* of possible branching patterns, which is by definition an unordered list of the integers $k \geq 1$ that determine the restricted exponent differences $1/k$. The following list of types is obtained:

$$(3.1) \quad (), (2), (3), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (4, 4).$$

The first type $()$ means no restrictions on the parameters of E_1 . We skipped the types (1) and (2, 2); they are considered in [32] as mentioned. The types are indicated by the exponent differences of E_1 in the first columns of Tables 1, 3, and

Exponent differences of the Heun equation	Deg. D	Branching pattern above singularities	Covering characterization, composition
$\alpha, \alpha, \alpha, 9\alpha$	12	$[2]_6 = [3]_4 = 9+1+1+1$	$H_1, 3_C \cdot 4$
$\alpha, \alpha, 2\alpha, 8\alpha$		$[2]_6 = [3]_4 = 8+2+1+1$	$H_2, F_{23}, F_{34}^*, 2 \cdot 2 \cdot 3$
$\alpha, \alpha, 3\alpha, 7\alpha$		$[2]_6 = [3]_4 = 7+3+1+1$	no covering, N_1
$\alpha, 2\alpha, 2\alpha, 7\alpha$		$[2]_6 = [3]_4 = 7+2+2+1$	no covering, N_2
$\alpha, \alpha, 4\alpha, 6\alpha$		$[2]_6 = [3]_4 = 6+4+1+1$	no covering, N_3
$\alpha, 2\alpha, 3\alpha, 6\alpha$		$[2]_6 = [3]_4 = 6+3+2+1$	$H_3, F_{27}, F_{33}^*, 3 \cdot 4, 4 \cdot 3$
$2\alpha, 2\alpha, 2\alpha, 6\alpha$		$[2]_6 = [3]_4 = 6+2+2+2$	no covering, N_4
$\alpha, \alpha, 5\alpha, 5\alpha$		$[2]_6 = [3]_4 = 5+5+1+1$	$H_4, F_{24}, F_{32}^*, 2_H \cdot 6$
$\alpha, 2\alpha, 4\alpha, 5\alpha$		$[2]_6 = [3]_4 = 5+4+2+1$	no covering, N_5
$\alpha, 3\alpha, 3\alpha, 5\alpha$		$[2]_6 = [3]_4 = 5+3+3+1$	no covering, N_6
$2\alpha, 2\alpha, 3\alpha, 5\alpha$		$[2]_6 = [3]_4 = 5+3+2+2$	no covering, N_7
$\alpha, 3\alpha, 4\alpha, 4\alpha$		$[2]_6 = [3]_4 = 4+4+3+1$	no covering, N_8
$2\alpha, 2\alpha, 4\alpha, 4\alpha$	10	$[2]_6 = [3]_4 = 4+4+2+2$	$H_5, F_{22}, F_{31}^*, 2 \cdot 3_C \cdot 2, 2 \times 2 \cdot 3$
$2\alpha, 3\alpha, 3\alpha, 4\alpha$		$[2]_6 = [3]_4 = 4+3+3+2$	no covering, N_9
$3\alpha, 3\alpha, 3\alpha, 3\alpha$		$[2]_6 = [3]_4 = 3+3+3+3$	$H_6, 3_C \cdot 4, 2_H \cdot 2_H \cdot 3_C$
$1/3, \alpha, \alpha, 8\alpha$		$[2]_5 = [3]_3+1 = 8+1+1$	H_7 , indecomposable
$1/3, \alpha, 2\alpha, 7\alpha$		$[2]_5 = [3]_3+1 = 7+2+1$	H_8, F_{21}, F_{28}^* , indecomposable
$1/3, \alpha, 3\alpha, 6\alpha$		$[2]_5 = [3]_3+1 = 6+3+1$	no covering, N_{10}
$1/3, 2\alpha, 2\alpha, 6\alpha$		$[2]_5 = [3]_3+1 = 6+2+2$	no covering, N_{11}
$1/3, \alpha, 4\alpha, 5\alpha$		$[2]_5 = [3]_3+1 = 5+4+1$	H_9, F_{19}, F_{29}^* , indecomposable
$1/3, 2\alpha, 3\alpha, 5\alpha$		$[2]_5 = [3]_3+1 = 5+3+2$	H_{10}, F_{26}, F_{30}^* , indecomposable
$1/3, 2\alpha, 4\alpha, 4\alpha$		$[2]_5 = [3]_3+1 = 4+4+2$	no covering, N_{12}
$1/3, 3\alpha, 3\alpha, 4\alpha$		$[2]_5 = [3]_3+1 = 4+3+3$	no covering, N_{13}
$1/2, \alpha, \alpha, 7\alpha$	9	$[2]_{4+1} = [3]_3 = 7+1+1$	H_{11} , indecomposable
$1/2, \alpha, 2\alpha, 6\alpha$		$[2]_{4+1} = [3]_3 = 6+2+1$	$H_{12}, F_{20}, F_{27}^*, 3 \cdot 3$
$1/2, \alpha, 3\alpha, 5\alpha$		$[2]_{4+1} = [3]_3 = 5+3+1$	H_{13}, F_{18}, F_{26}^* , indecomposable
$1/2, 2\alpha, 2\alpha, 5\alpha$		$[2]_{4+1} = [3]_3 = 5+2+2$	no covering, N_{14}
$1/2, \alpha, 4\alpha, 4\alpha$		$[2]_{4+1} = [3]_3 = 4+4+1$	no covering, N_{15}
$1/2, 2\alpha, 3\alpha, 4\alpha$		$[2]_{4+1} = [3]_3 = 4+3+2$	$H_{14}, F_{25}, F_{25}^*, 3 \cdot 3$
$1/2, 3\alpha, 3\alpha, 3\alpha$		$[2]_{4+1} = [3]_3 = 3+3+3$	no covering, N_{16}
$2/3, \alpha, \alpha, 6\alpha$	8	$[2]_4 = [3]_{2+2} = 6+1+1$	$H_{15}, F_{14}, 2 \cdot 4$
$2/3, \alpha, 2\alpha, 5\alpha$		$[2]_4 = [3]_{2+2} = 5+2+1$	H_{16}, F_{17} , indecomposable
$2/3, \alpha, 3\alpha, 4\alpha$		$[2]_4 = [3]_{2+2} = 4+3+1$	no covering, N_{17}
$2/3, 2\alpha, 2\alpha, 4\alpha$		$[2]_4 = [3]_{2+2} = 4+2+2$	no covering, N_{18}
$2/3, 2\alpha, 3\alpha, 3\alpha$		$[2]_4 = [3]_{2+2} = 3+3+2$	$H_{17}, F_{13}, 2 \cdot 4$
$1/3, 1/3, \alpha, 7\alpha$		$[2]_4 = [3]_{2+1+1} = 7+1$	H_{18} , indecomposable
$1/3, 1/3, 2\alpha, 6\alpha$		$[2]_4 = [3]_{2+1+1} = 6+2$	$H_{19}, F_{16}, F_{21}^*, 4_B \cdot 2, 2 \cdot 4$
$1/3, 1/3, 3\alpha, 5\alpha$		$[2]_4 = [3]_{2+1+1} = 5+3$	no covering, N_{19}
$1/3, 1/3, 4\alpha, 4\alpha$		$[2]_4 = [3]_{2+1+1} = 4+4$	$H_{20}, F_{15}, F_{20}^*, 4 \cdot 2, 2_H \cdot 4_A$
$1/2, 1/3, \alpha, 6\alpha$	7	$[2]_{3+1} = [3]_{2+1} = 6+1$	H_{21} , indecomposable
$1/2, 1/3, 2\alpha, 5\alpha$		$[2]_{3+1} = [3]_{2+1} = 5+2$	H_{22}, F_{11}, F_{18}^* , indecomposable
$1/2, 1/3, 3\alpha, 4\alpha$		$[2]_{3+1} = [3]_{2+1} = 4+3$	H_{23}, F_{12}, F_{19}^* , indecomposable

TABLE 2. Possible branching patterns for pull-back transformations from $E(1/2, 1/3, \alpha)$ to a Heun equation, of degree $D \geq 7$.

Exponent differences		Deg. D	Branching pattern above singularities	Covering characterization, composition
hyperg.	Heun			
$1/2, 1/3, \alpha$	$1/3, 2/3, \alpha, 5\alpha$	6	$[2]_3 = [3]_{1+2+1} = 5+1$	H_{24}, F_9 , indecomposable
	$1/3, 2/3, 2\alpha, 4\alpha$		$[2]_3 = [3]_{1+2+1} = 4+2$	$H_{25}, F_{10}, 3 \cdot 2$
	$1/3, 2/3, 3\alpha, 3\alpha$		$[2]_3 = [3]_{1+2+1} = 3+3$	no covering, N_{20}
	$1/3, 1/3, 1/3, 6\alpha$	5	$[2]_3 = [3]_{1+1+1+1} = 6$	$H_{38}, 3_C \cdot 2$
	$1/2, 1/2, \alpha, 5\alpha$		$[2]_{2+1+1} = [3]_2 = 5+1$	H_{26} , indecomposable
	$1/2, 1/2, 2\alpha, 4\alpha$		$[2]_{2+1+1} = [3]_2 = 4+2$	$H_{27}, F_7, F_{13}^*, 2 \cdot 3$
	$1/2, 1/2, 3\alpha, 3\alpha$		$[2]_{2+1+1} = [3]_2 = 3+3$	$H_{28}, F_6, F_{12}^*, 2_H \cdot 3_C$
	$1/2, 2/3, \alpha, 4\alpha$		$[2]_{2+1} = [3]_{1+2} = 4+1$	H_{29}, F_8 , indecomposable
	$1/2, 2/3, 2\alpha, 3\alpha$		$[2]_{2+1} = [3]_{1+2} = 3+2$	H_{30}, F_5 , indecomposable
	$1/2, 1/3, 1/3, 5\alpha$		$[2]_{2+1} = [3]_{1+1+1} = 5$	H_{37} , indecomposable
	$1/2, 1/2, 1/3, 4\alpha$	4	$[2]_{1+1+1} = [3]_{1+1} = 4$	H_{36} , indecomposable
$1/2, 1/4, \alpha$	$\alpha, \alpha, \alpha, 5\alpha$	8	$[2]_4 = [4]_2 = 5+1+1+1$	no covering, N_{21}
	$\alpha, \alpha, 2\alpha, 4\alpha$		$[2]_4 = [4]_2 = 4+2+1+1$	$H_{40}, F_9', F_{24}^*, 2 \cdot 2 \cdot 2$
	$\alpha, \alpha, 3\alpha, 3\alpha$		$[2]_4 = [4]_2 = 3+3+1+1$	$H_{20}, F_8', F_{23}^*, 4 \cdot 2, 2_H \cdot 4_A$
	$\alpha, 2\alpha, 2\alpha, 3\alpha$	6	$[2]_4 = [4]_2 = 3+2+2+1$	no covering, N_{22}
	$2\alpha, 2\alpha, 2\alpha, 2\alpha$		$[2]_4 = [4]_2 = 2+2+2+2$	$H_{41}, F_7', F_{22}^*, 2 \times 2 \times 2$
	$1/2, \alpha, \alpha, 4\alpha$		$[2]_3 = [4]_{1+2} = 4+1+1$	no covering, N_{23}
	$1/2, \alpha, 2\alpha, 3\alpha$		$[2]_3 = [4]_{1+2} = 3+2+1$	$H_{25}, F_6', F_{15}^*, 3 \cdot 2$
	$1/2, 2\alpha, 2\alpha, 2\alpha$		$[2]_3 = [4]_{1+2} = 2+2+2$	no covering, N_{24}
	$1/4, 1/4, \alpha, 5\alpha$	5	$[2]_3 = [4]_{1+1+1} = 5+1$	H_{42} , indecomposable
	$1/4, 1/4, 2\alpha, 4\alpha$		$[2]_3 = [4]_{1+1+1} = 4+2$	no covering, N_{23}
	$1/4, 1/4, 3\alpha, 3\alpha$		$[2]_3 = [4]_{1+1+1} = 3+3$	$H_{43}, F_{14}', F_{14}^*, 2_H \cdot 3$
	$1/2, 1/4, \alpha, 4\alpha$		$[2]_{2+1} = [4]_{1+1} = 4+1$	H_{44} , indecomposable
	$1/2, 1/4, 2\alpha, 3\alpha$		$[2]_{2+1} = [4]_{1+1} = 3+2$	$H_{29}, F_{11}', F_{10}^*$, indecomposable
	$1/2, 1/2, \alpha, 3\alpha$	4	$[2]_{1+1+1} = [4]_1 = 3+1$	H_{36} , indecomposable
	$1/2, 1/2, 2\alpha, 2\alpha$		$[2]_{1+1+1} = [4]_1 = 2+2$	$H_{35}, F_{10}', F_7^*, 2_H \cdot 2$
$1/2, 1/5, \alpha$	$1/5, \alpha, \alpha, 4\alpha$	6	$[2]_3 = [5]_{1+1} = 4+1+1$	H_{42} , indecomposable
	$1/5, \alpha, 2\alpha, 3\alpha$		$[2]_3 = [5]_{1+1} = 3+2+1$	$H_{24}, F_{15}', F_{16}^*$, indecomposable
	$1/5, 2\alpha, 2\alpha, 2\alpha$		$[2]_3 = [5]_{1+1} = 2+2+2$	no covering, N_{25}
	$1/2, \alpha, \alpha, 3\alpha$	5	$[2]_{2+1} = [5]_1 = 3+1+1$	H_{37} , indecomposable
	$1/2, \alpha, 2\alpha, 2\alpha$		$[2]_{2+1} = [5]_1 = 2+2+1$	$H_{45}, F_{12}', F_{11}^*$, indecomposable
$1/2, 1/6, \alpha$	$\alpha, \alpha, \alpha, 3\alpha$	6	$[2]_3 = [6]_1 = 3+1+1+1$	$H_{38}, 3_C \cdot 2$
	$\alpha, \alpha, 2\alpha, 2\alpha$		$[2]_3 = [6]_1 = 2+2+1+1$	$H_{39}, F_{13}', F_{17}^*, 3 \cdot 2, 2_H \cdot 3$
$1/3, 1/3, \alpha$	$\alpha, \alpha, \alpha, 3\alpha$	6	$[3]_2 = [3]_2 = 3+1+1+1$	no covering, N_{26}
	$\alpha, \alpha, 2\alpha, 2\alpha$		$[3]_2 = [3]_2 = 2+2+1+1$	$H_{28}, 2 \cdot 3_C$
	$1/3, 1/3, \alpha, 3\alpha$	4	$[3]_{1+1} = [3]_{1+1} = 3+1$	H_{46}, F_4'', F_9^* , indecomposable
	$1/3, 1/3, 2\alpha, 2\alpha$		$[3]_{1+1} = [3]_{1+1} = 2+2$	H_{47}, F_3'', F_8^* , indecomposable
$1/3, 1/4, \alpha$	$1/3, \alpha, \alpha, 2\alpha$	4	$[3]_{1+1} = [4]_1 = 2+1+1$	H_{36} , indecomposable
$1/4, 1/4, \alpha$	$\alpha, \alpha, \alpha, \alpha$	4	$[4]_1 = [4]_1 = 1+1+1+1$	$H_{48}, 2_H \cdot 2$

TABLE 3. The other possible branching patterns of Gauss-to-Heun transformations with one free parameter.

Id	Deg.	Branching pattern	A rational expression for $\varphi(x)$
H_1	12	$[2]_6 = [3]_4 = 9+1+1+1$	$64x^3(x^3-1)^3/(8x^3-9)$
H_2		$[2]_6 = [3]_4 = 8+2+1+1$	$27x^2(x^2-4)/4(x^4-4x^2+1)^3$
H_3		$[2]_6 = [3]_4 = 6+3+2+1$	$27(x-1)^3(2x-3)^2(x+3)/4x^3(x^3-6x+6)^3$
H_4		$[2]_6 = [3]_4 = 5+5+1+1$	$1728x^5(x^2-11x-1)/(x^4-12x^3+14x^2+12x+1)^3$
H_5		$[2]_6 = [3]_4 = 4+4+2+2$	$27x^4(x^2-1)^2/4(x^4-x^2+1)^3$
H_6		$[2]_6 = [3]_4 = 3+3+3+3$	$-64x^3(x^3-1)^3/(8x^3+1)^3$
H_7	10	$[2]_5 = [3]_3+1 = 8+1+1$	$-4(x+2)(x^3+3x+2)^3/27(3x^2-2x+11)$
H_8		$[2]_5 = [3]_3+1 = 7+2+1$	$4(x+4)(x^3-6x-2)^3/27(3x+4)^2(4x-11)$
H_9		$[2]_5 = [3]_3+1 = 5+4+1$	$-(8x-1)(8x^3+87x^2+96x-64)^3/2^33^{12}x^4(x+10)$
H_{10}		$[2]_5 = [3]_3+1 = 5+3+2$	$-(x-3)(81x^3-9x^2-53x-27)^3/2^{14}3^3x^3(9x+5)^2$
H_{11}	9	$[2]_4+1 = [3]_3 = 7+1+1$	$4(x^3+4x^2+10x+6)^3/27(4x^2+13x+32)$
H_{12}		$[2]_4+1 = [3]_3 = 6+2+1$	$27x^2(x-3)/4(x^3-3x^2+1)^3$
H_{13}		$[2]_4+1 = [3]_3 = 5+3+1$	$-25(5x^3+45x^2+39x-25)^3/2^{14}3^3x^3(3x+25)$
H_{14}		$[2]_4+1 = [3]_3 = 4+3+2$	$27x^3(3x-4)^2/4(x^3-3x-4)^3$
H_{15}	8	$[2]_4 = [3]_2+2 = 6+1+1$	$64x^2(x^2-1)^3/(8x^2-9)$
H_{16}		$[2]_4 = [3]_2+2 = 5+2+1$	$-4x^2(x^2+8x+10)^3/27(2x+1)^2(4x+27)$
H_{17}		$[2]_4 = [3]_2+2 = 3+3+2$	$-64x^2(x^2-1)^3/(8x^3+1)^3$
H_{18}		$[2]_4 = [3]_2+1+1 = 7+1$	$-(x^2-13x+49)(x^2-5x+1)^3/2^63^3x$
H_{19}		$[2]_4 = [3]_2+1+1 = 6+2$	$-64x^2/(x^2-1)^3(x^2-9)$
H_{20}		$[2]_4 = [3]_2+1+1 = 4+4$	$16x^3(2x+1)(x-4)/(x^2-2x-2)^4$
H_{21}	7	$[2]_3+1 = [3]_2+1 = 6+1$	$4(x-1)((1+2\omega)x^2-3x-\omega)^3/(4-(1+3\omega)x)$
H_{22}		$[2]_3+1 = [3]_2+1 = 5+2$	$x(2x^2-35x+140)^3/108(14x-125)^2$
H_{23}		$[2]_3+1 = [3]_2+1 = 4+3$	$-(x+27)(16x^2+80x-243)^3/2^23^37^7x^3$
H_{24}	6	$[2]_3 = [3]_1+2+1 = 5+1$	$-x^2(x-3)(x+5)^3/64(3x+16)$
H_{25}		$[2]_3 = [3]_1+2+1 = 4+2$	$-4x^3(x-1)^2(x+2)/(3x-2)^2$
H_{26}		$[2]_2+1+1 = [3]_2 = 5+1$	$1728x/(x^2+10x+5)^3$
H_{27}		$[2]_2+1+1 = [3]_2 = 4+2$	$27x^2/4(x^2-1)^3$
H_{28}		$[2]_2+1+1 = [3]_2 = 3+3$	$(x^2+6x-3)^3/(x^2-6x-3)^3$
H_{29}	5	$2+2+1 = 3+2 = 4+1$	$4x^2(x+5)^3/27(5x+27)$
H_{30}		$2+2+1 = 3+2 = 3+2$	$x^2(9x-5)^3/4(5x-1)^2$
H_{31}	4	$2+2 = 2+2 = 2+2$	$-4x^2/(x^2-1)^2$
H_{32}	2	$2 = 2 = 1+1$	x^2
H_{33}	3	$3 = 3 = 1+1+1$	x^3
H_{34}		$3 = 2+1 = 2+1$	$x(4x-3)^2$
H_{35}	4	$4 = 2+2 = 2+1+1$	$4x^2(1-x^2)$
H_{36}		$4 = 3+1 = 2+1+1$	$-x^3(3x+4)$
H_{37}	5	$5 = 2+2+1 = 3+1+1$	$(x^2+11x+64)(x+3)^3/2^63^3$
H_{38}	6	$[2]_3 = 6 = 3+1+1+1$	$4x^3(1-x^3)$
H_{39}	6	$[2]_3 = 6 = 2+2+1+1$	$x^2(4x^2-3)^2$
H_{40}	8	$[2]_4 = [4]_2 = 4+2+1+1$	$4x^2(x^2-2)/(x^2-1)^4$
H_{41}		$[2]_4 = [4]_2 = 2+2+2+2$	$-4x^4/(x^4-1)^2$
H_{42}	6	$[2]_3 = 4+1+1 = 5+1$	$-(x-1)^4(x^2-6x+25)/256x$
H_{43}		$[2]_3 = 4+1+1 = 3+3$	$27x(x-4)/4(x^2-4x+1)^3$
H_{44}	5	$4+1 = 4+1 = 2+2+1$	$x(x-1-2i)^4/((1+2i)x-1)^4$
H_{45}		$5 = 2+2+1 = 2+2+1$	$x(x^2-5x+5)^2/4$
H_{46}	4	$3+1 = 3+1 = 3+1$	$x^3(x+2)/(2x+1)$
H_{47}		$3+1 = 3+1 = 2+2$	$64x(x-1)^3/(8x-9)$
H_{48}		$4 = 4 = 1+1+1+1$	x^4

TABLE 4. The Belyi coverings appearing in Gauss-to-Heun pull-backs, up to Möbius transformations.

the whole Table 2 is devoted to the type $(2, 3)$. The entries of different types are separated by horizontal lines. The pull-back degree is given in the third columns of Tables 1, 3, and the second column of Table 2. The maximal degree is 12. It occurs for the type $(2, 3)$ only.

Step 2 is determination of possible branching patterns. The method is explained in §3.2. The result is presented by the fourth columns of Tables 1, 3, and the third column of Table 2. Generally, we indicate a branching pattern by an (unordered) list of three unordered partitions of its degree D , separated by the equality signs. The partitions specify the branching indices in each of the three branching fibers of a Belyi covering. Besides, we use the abbreviation $[k]_n$ for a partition block $k + \dots + k$ (n times). In Tables 1, 2, 3, the symbol $[k]_n$ specifically means presence of n points of E_2 with the branching order k above a singular point of E_1 with the exponent difference $1/k$. By part (c) of Lemma 2.1, each of the n points will be either ordinary or an irrelevant singularity for E_2 . By the described convention, the branching patterns for pull-back transformations with M free parameters have $3 - M$ numbers (i.e., branching orders) inclosed in square brackets, and exactly 4 non-bracketed numbers representing the 4 singular points of E_2 .

In total, we get a list of 89 branching patterns, though some of the patterns differ only by the square-brackets specification of ordinary points of E_2 . For example, two degree 3 branching patterns in Table 1 are the same, leading to the same cubic covering H_{34} (identified in the last column). The exponent differences of E_2 are determined by E_1 and the branching pattern, and are given by the second columns of Tables 1, 3 and the first column of Table 2.

Step 3 is computation of the Belyi coverings $\varphi: \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$. Generally, computation of Belyi maps with a given branching pattern is a difficult problem. However, the maximal degree implied by the possible branching patterns is just 12. With the aid of modern computer algebra systems this problem is tractable for coverings of degree 12 or less, even using a straightforward Ansatz method with undetermined coefficients. Most of the Belyi maps are actually known in the literature, if only because the Belyi maps of the type $(2, 3)$ occur in Herfurtnier's list [10] of elliptic surfaces with four singular fibers. Specifically, the $\mathcal{J}(X, Y)$ -expressions in [10, Table 3] are homogeneous expressions of the Belyi maps H_1, \dots, H_{38} up to Möbius transformations. Moreover, the same coverings basically appear in pull-backs between hypergeometric equations, because a free parameter can always be specialized so to reduce the Heun equation E_2 to a hypergeometric (or simpler) equation.

The computational issues of Step 3 are discussed in §3.2. Complementarily, §5 presents an elegant approach to show non-existence of Belyi maps with many branching patterns. The full list of computed Belyi maps is given in Table 4, and further commented in §4. The last columns of Tables 1, 2, 3 identify the Belyi map for each possible pull-back transformation. These columns also specify the *Coxeter decompositions* [7] and *divisible tilings* [5] for the Schwarz maps associated to the pulled-back Heun's equation E_2 (by various F -numbers, as explained in §4.3), and describe composite transformations by product expressions indicating degrees of occurring indecomposable transformations. The product notation has to be followed from right to left to trace the composition from the starting hypergeometric equation. The factor 2_H denotes quadratic Heun-to-Heun transformation (2.7). Here is the meaning of other indexed degrees: 3_C denotes the cyclic covering H_{33} with the branching pattern $3 = 1 + 1 + 1$, while 4_A and 4_B stand for the

coverings H_{36} ($4 = 3+1 = 2+1+1$) and H_{46} ($3+1 = 3+1 = 3+1$), respectively. The unindexed numbers 3 and 4 denote the frequent coverings H_{34} ($3 = 2+1 = 2+1$) and H_{47} ($3+1 = 3+1 = 2+2$), respectively. In any composition, there is exactly one factor representing an indecomposable Gauss-to-Heun transformation; it is the first one from the left which is not 2_H . The other factors to the right represent pull-backs between hypergeometric equations. The notation 2×2 indicates a composition of quadratic transformations that can be realized in multiple ways, possibly including 2_H ; see (4.3) below for the most typical example. The compositions are considered more thoroughly in §4.4 and in [30, Appendix B].

There are 27 different branching patterns for which there is no Belyi map. The non-existence in all these cases can be elegantly shown by considering implied (but not possible) pull-back transformations between Fuchsian equations, as explained in §5. The indexed N -notation refers to Table 5 below. For each branching pattern except two leading to H_{21} and H_{44} , there is at most one covering (and one pull-back) up to Möbius transformations. The coverings H_{21} and H_{44} are defined, respectively, over $\mathbb{Q}(\omega)$ and $\mathbb{Q}(i)$. In either of these cases, we actually have a complex-conjugated pair of Belyi coverings. Table 4 lists 48 different coverings, though H_{21} and H_{44} should be properly counted twice. It is instructive to compare the branching pattern and the orders of vertices and cells of the *dessins d'enfant* in Figure 1. In total, we count 61 parametric pull-backs among the entries of Tables 1, 2, 3. Of them, 28 are composite. Evidently, some of the 48 coverings appear in more than one pull-back. Accordingly, the symbol $[k]_n$ in Table 4 merely indicates presence of n points of branching order k in the same fiber. The coverings H_{20} , H_{24} , H_{25} , H_{28} , H_{29} , H_{34} , H_{35} , H_{37} , H_{38} , H_{42} , H_{47} appear twice in Tables 1, 2, 3, while H_{36} three times.

Step 4 is derivation of identities between standard ${}_2F_1(z)$ and $\text{Hn}(x)$ solutions of the related hypergeometric and Heun equations, with $z = \varphi(x)$. This gives *Heun-to-hypergeometric reduction* formulas, expressing found Heun functions in terms of the better understood Gauss hypergeometric functions. This final step is comprehensively considered in the parallel paper [30] by the same authors. In particular, [30, §3] explains the technical issue of choosing the gauge prefactor $\theta(x)$ in pull-back transformations (1.4). The transformations without a prefactor (i.e., $\theta(x) = 1$) are classified by Maier in [16]. The branching patterns for these pull-backs typically have a fiber with just one point, and that point is a singularity for E_2 . There are 7 of these pull-back transformations. Their type is $()$, (2) , (3) or $(2,3)$, and the coverings are numbered consequently from H_{32} to H_{38} . Formulas without a prefactor arise from the transformations of Tables 1, 2 realized by these coverings, except for the type (3) transformation with the covering H_{34} . The well-known quadratic transformation (2.8) is described at the beginning of this article.

Hereby we complete the description of four classification steps. At the same time, we explained the results and notation in Tables 1–4. The next two subsections give a methodological proof of Steps 1 and 2. Section 4 discusses computational issues of Step 3, composite coverings, and relations of the recorded transformations to the Herfurtners list [10] of elliptic surfaces and Felikson's list [7] of Coxeter decompositions. Section 5 describes the elegant approach of proving non-existence of Belyi coverings with certain branching patterns, and applies it to Tables 1–3 and the Miranda-Persson list [18] of degree 24 branching patterns.

3.1. Step 1: Possible restricted exponent differences and degree. We are looking for the Belyi coverings $\varphi: \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$ that pull-back a hypergeometric equation E_1 to Heun's equation E_2 . We assume that E_1 is not specifically of the form $E(1, \alpha, \beta)$ or $E(1/2, 1/2, \alpha)$, because then either it has a logarithmic singularity (if $\beta \neq \pm\alpha$ by [29, Lemma 5.1]; it would not contribute extra non-singular points above $\{0, 1, \infty\} \subset \mathbb{P}_z^1$ necessary for new cases of Gauss-to-Heun pull-backs), or it has cyclic (if $\beta = |\alpha|$) or dihedral monodromy as explored in [32].

We restrict $m \in \{0, 1, 2\}$ exponent differences of the general hypergeometric equation (1.1) to the reciprocals of integers $k > 1$, and look for particular cases when part (c) of Lemma 2.1 allows enough non-singular points above $\{0, 1, \infty\} \subset \mathbb{P}_z^1$. The degree of φ is denoted by D .

First, assume that $m = 0$. This puts no restriction on the exponent differences of E_1 , so generally all points above $z = 0, 1, \infty$ will be singularities of E_2 . There will be exactly $D + 2$ singular points by Lemma 2.2, and we wish the transformed equation to be Heun's. Hence $D + 2 \leq 4$, so that $D \leq 2$. If $D = 1$ then φ is a Möbius transformation and does not alter the number of singular points, hence E_2 will have only three. If $D = 2$, then φ is a quadratic covering with the branching pattern $2 = 2 = 1 + 1$. The pull-back is then the well-known quadratic transformation (2.8), applicable to any hypergeometric equation. We do not need to consider quadratic transformations ($D = 2$) subsequently, nor the case $m = 0$ in more detail in the other steps.

For $m \in \{1, 2\}$, the number of non-singular points above the restricted singularities of E_1 must be at least $(D + 2) - 4 = D - 2$.

If $m = 1$, we allow two free parameters. We restrict just one exponent difference of E_1 to equal $1/k$, with integer $k > 1$. The pulled-back equation E_2 will have at most $\lfloor D/k \rfloor$ ordinary points above $\{0, 1, \infty\} \subset \mathbb{P}_z^1$ by Lemma 2.1, and one must have

$$(3.2) \quad \left\lfloor \frac{D}{k} \right\rfloor \geq D - 2.$$

This leads to the Diophantine inequality

$$(3.3) \quad \frac{2}{D} + \frac{1}{k} \geq 1.$$

For $k > 1$ and $D > 2$, we have the following possibilities:

$$(3.4) \quad (k, D) \in \{(2, 3), (2, 4), (3, 3)\}.$$

The resulting branching patterns are of types (2), (3), according to k .

If $m = 2$, we allow one free parameter. Suppose that the restricted exponent differences of E_1 equal $1/k, 1/\ell$, where k, ℓ are integers. We assume $1 < k \leq \ell \leq D$ without loss of generality; the last inequality allows actual utilization of the restriction $1/\ell$. The transformed equation has at most $\lfloor D/k \rfloor + \lfloor D/\ell \rfloor$ ordinary points above $\{0, 1, \infty\} \subset \mathbb{P}_z^1$. Similarly to the above, one must have

$$(3.5) \quad \left\lfloor \frac{D}{k} \right\rfloor + \left\lfloor \frac{D}{\ell} \right\rfloor \geq D - 2,$$

which leads to the weaker Diophantine inequality

$$(3.6) \quad \frac{2}{D} + \frac{1}{k} + \frac{1}{\ell} \geq 1.$$

Discarding $k = \ell = 2$, we get the following possibilities for k, ℓ and for the upper bound D_{\max} on the degree D :

(3.7)

$$(k, \ell, D_{\max}) \in \{(2, 3, 12), (2, 4, 8), (2, 5, 6), (2, 6, 6), (3, 3, 6), (3, 4, 4), (4, 4, 4)\}.$$

The resulting branching patterns are the seven types $(2, 3)$, $(2, 4)$, $(2, 5)$, $(2, 6)$, $(3, 3)$, $(3, 4)$, $(4, 4)$, respectively.

3.2. Step 2: Possible branching patterns. Here we look at each possible type and degree, and determine all branching patterns fitting them. The constraint that there must be at least $D-2$ ordinary points above $\{0, 1, \infty\} \subset \mathbb{P}_z^1$ will usually require taking the number of ordinary points above the points with restricted exponent differences is *maximal*, i.e., equal to $\lfloor D/k \rfloor$ or $\lfloor D/\ell \rfloor$.

The *a priori* possible branching patterns for the case $m = 1$ are straightforward to determine. They are listed in the fourth column of Table 1. That table is comparable to [19, Table 1b].

In the case $m = 2$, we start with the coverings of the type $(2, 3)$ of the maximal degree $D = 12$, as in Table 2. There must be $12 - 2 = 10$ ordinary points above the two singular points of $E(1/2, 1/3, \alpha)$ with exponent differences $1/2$ or $1/3$; all x -points in these two fibers must be ordinary, as $\lfloor 12/2 \rfloor + \lfloor 12/3 \rfloor = 10$. The third fiber is a partition of 12 with 4 parts. There are 15 such partitions, and they are all listed in the third column of Table 2. Next, there are no transformations of degree 11, because $11 - 2 > \lfloor 11/2 \rfloor + \lfloor 11/3 \rfloor$ and there would not be enough ordinary points in the two fibers. In a similar way, the pull-back coverings of degree $D = 10, 9, 8$ or 7 must have the maximal number of ordinary points in the two restricted fibers; and all branching patterns consistent with this constraint are listed. The branching patterns of type $(2, 3)$ continue in Table 3. The degrees $D = 6, 4$ require less than $\lfloor D/2 \rfloor + \lfloor D/3 \rfloor$ ordinary points in the restricted fibers, and there is some choice of how to split a bracketed number $[2]$ or $[3]$ into a pair of non-bracketed numbers, though at least one bracketed number must remain in the two restricted fibers. For $D = 5$, there is a choice of splitting (or not splitting) the number 2 in the $[3]$ fiber. In total, we get 53 branching patterns of the type $(2, 3)$, all different.

The other types $(2, 4)$, $(2, 5)$, $(2, 6)$, $(3, 3)$, $(3, 4)$, $(4, 4)$ similarly give less numerous sets of branching patterns, some of them coinciding mutually or with previously encountered ones.

4. THE BELYI COVERINGS

First, this section briefly explains computation of Belyi maps and utilizing specialization of parametric Gauss-to-Heun transformations to transformations between hypergeometric equations. In §§4.2–4.3, we explain how the H -numbering of Table 4 comes partly from an algebraic-geometric classification of Herfurtnier [10], and clarify the various F -numbers in the last columns of Tables 1–3 as representing *Coxeter decompositions* of Felikson [7] and *divisible tilings* of [5]. Lastly, in §4.4 we examine the composite transformations among our results.

4.1. Computational issues. To compute the Belyi maps $\varphi: \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$ with a given branching pattern means to find all rational functions $\varphi(x)$ such that the numerators of $\varphi(x)$, $1 - \varphi(x)$ and the denominator of $\varphi(x)$ factor according to the branching pattern. A straightforward Ansatz method with undetermined coefficients can be used for low degree coverings. Modern computer algebra systems

(such as **Maple** and **Mathematica**) can handle the resulting systems of algebraic equations easily if the degree of $\varphi(x)$ is 12 or less. More cannily, one can consider factorization of the numerators of the logarithmic derivatives of $\varphi(x)$ and $\varphi(x) - 1$ as in [26, §3]. For example, to determine H_1 , one is looking for a constant c and monic polynomials P, Q, R of degree 4, 3, 6, respectively, such that $\varphi(x) = cP^3/Q$ and $\varphi(x) - 1 = cR^2/Q$. To find these polynomials, one considers

$$(4.1) \quad \frac{\varphi'(x)}{\varphi(x)} = \frac{3P'}{P} - \frac{Q'}{Q} \stackrel{!}{=} \frac{9R'}{PQ}, \quad \frac{(\varphi(x) - 1)'}{\varphi(x) - 1} = \frac{2R'}{R} - \frac{Q'}{Q} \stackrel{!}{=} \frac{9P^2}{RQ}.$$

Zeros of the derivatives are the branching points other than in the denominators, and the factor 9 is determined by local consideration at $x = \infty$. The whole polynomial R can be eliminated symbolically using the first identification, and the resulting equation system for the undetermined coefficients of P, Q is rather transparent. In general, a covering with a given branching pattern may not exist, or there may be several Belyi maps (up to Möbius equivalence) or even several $\overline{\mathbb{Q}}/\mathbb{Q}$ -Galois orbits of Belyi maps with the same branching pattern. The Galois action on the Belyi maps and their *dessins d'enfant* is of primary interest to Grothendieck's theory [22], [24].

Less demanding, one may notice that the free parameter of our Gauss-to-Heun transformations can be specialized so that the pulled-back Heun equation has actually less than 4 singular points. Therefore, the Belyi coverings must appear in the classification [29] of Gauss-to-Gauss transformations in principle, though there are a few infinite families of those transformations (for degenerate, dihedral, algebraic or elliptic hypergeometric functions). In particular, each branching pattern of Table 1 can be found in [29, Table 1], except for $2+2=2+2=2+2$ which corresponds to the transformation $E(1/2, 1/2, \alpha) \stackrel{4}{\leftarrow} E(1, 2\alpha, 2\alpha)$ briefly mentioned in [29, p. 161]. The branching patterns of Tables 2 and 3 (with $m = 1$ free parameter) can be handled similarly, yielding reductions of one-parameter Gauss-to-Heun transformations to zero-parameter pull-backs between hypergeometric functions. For example, the covering H_{27} implies the hypergeometric transformations $E(1/2, 1/3, 1/2) \stackrel{6}{\leftarrow} E(1/2, 1/2, 2)$ and $E(1/2, 1/3, 1/4) \stackrel{6}{\leftarrow} E(1/2, 1/2, 1/2)$. These specializations reductions are possible whenever there is a branching point with a free exponent difference. Among the relevant branching patterns, only the last one ($4 = 4 = 1+1+1+1$) in Table 3 does not satisfy this condition. But even it represents a nominally hypergeometric transformation, namely $E(\alpha, \alpha, 1) \stackrel{4}{\leftarrow} E(4\alpha, 4\alpha, 1)$. Section 5 gives more details for obtaining the list of Gauss-to-Heun pull-backs from the classification in [29]. In particular, the non-unique coverings H_{21} and H_{44} come from Lemma 5.3.

4.2. The Herfurtnier classification. Pull-back transformations from hypergeometric equations of the form $E(1/2, 1/3, \alpha)$ to Heun equations have a close relation to elliptic surfaces over $\mathbb{C}(x)$ with 4 singular fibers [10, 19]. The Belyi coverings $z = \varphi(x)$ that induce these transformations appear as j -invariants of the elliptic surfaces, with z equal to $\mathcal{J} := j/1728$, the traditional Klein j -invariant.

The elliptic surfaces with 4 singular fibers are classified by Herfurtnier [10]. His article lists 50 configurations of singular fibers which give such elliptic surfaces, and for each configuration, supplies a formula $\mathcal{J} = \mathcal{J}(X, Y)$ which is a projectivized version of $z = \varphi(x)$, up to a Möbius transformation of x and a permutation of $z = 0, 1, \infty$. Heun equations arise from 38 of his 50 cases, as Movasati and Reiter [19]

recently observed. We adopt the enumeration of [19, Table 1], and denote these 38 Belyi coverings of Herfurtner, which were not originally numbered, by H_1 to H_{38} . The ordering is by degree in two ranges, as evident in Table 4: decreasing in the range H_1, \dots, H_{31} , and increasing in the range H_{32}, \dots, H_{38} .

By examining Table 2 and the upper part of Table 3, one finds that the coverings H_1, \dots, H_{30} and H_{36}, \dots, H_{38} induce Gauss-to-Heun pull-backs of the type $(2, 3)$ with one free parameter. These transformations use each of these 34 coverings exactly once, and no other coverings appear. The ordering by decreasing degree make the H -numbers appear ordered in Table 2, and almost ordered in the upper part of Table 3. By examining Table 1, one finds Herfurtner's coverings H_{31}, \dots, H_{35} (with H_{34} appearing twice) and a “new” covering H_{47} . The covering H_{47} cannot pull-back $E(1/2, 1/3, \alpha)$ to a Fuchsian equation with exactly 4 singularities. The pattern $[3]_1 = 2+1 = 2+1$ for H_{34} cannot be refined to such a pull-back from $E(1/2, 1/3, \alpha)$ either, but this is possible for the other H_{34} parsing $[2]_1 + 1 = 2+1 = 3$. This explains why H_{34} appears in Herfurtner's list once.

Some of Herfurtner's coverings additionally induce one-parameter Gauss-to-Heun transformations of types $(2, 4)$, $(2, 5)$, etc., as evident in Table 3. But 10 extra coverings appear in the latter sections of that table; they have no interpretation in terms of elliptic surfaces. We denote them H_{39}, \dots, H_{48} , ordered somewhat arbitrarily in the lower part of Table 4. The covering H_{47} induces transformations of the types (2) and $(3, 3)$.

4.3. Coxeter decompositions. Recall that a *Schwarz map* for an second order differential equation in the complex domain is a map $\mathbb{C} \rightarrow \mathbb{C}$ defined as the ratio of a pair of independent solutions of the differential equation [3]. Consider a hypergeometric equation with real exponent differences (α, β, γ) satisfying $0 \leq \alpha, \beta, \gamma < 1$. The image of the upper half plane under its Schwarz map is a curvilinear *Schwarz triangle*; the sides are line or circle segments, and the angles are equal to $\pi\alpha, \pi\beta, \pi\gamma$. Similarly, consider a Heun equation with real exponent differences $(\alpha, \beta, \gamma, \delta)$ satisfying $0 \leq \alpha, \beta, \gamma, \delta < 1$. The image of the upper half plane under its Schwarz map is a curvilinear *Schwarz quadrangle*, with the same kind of sides, and angles are equal to $\pi\alpha, \pi\beta, \pi\gamma, \pi\delta$.

It was noticed by Hodgkinson [11, 12] that if the covering $\varphi(x)$ of a pull-back transformation between hypergeometric equations is defined over \mathbb{R} , the analytic continuations of their solutions according to the Schwarz reflection principle are compatible. In consequence, the covering φ (of degree D , say) will induce a subdivision of a Schwarz triangle of the pulled-back hypergeometric equation into D Schwarz triangles of the original hypergeometric equation. Examples of such subdivisions are given in [26, Figure 1].

Similarly, suppose we have a Gauss-to-Heun transformation defined over \mathbb{R} . In particular, the fourth singular point $x = t$ is real. Then the analytic continuations of the hypergeometric and Heun solutions according to the Schwarz reflection principle are compatible, and the covering φ (of degree D) will induce a subdivision of a Schwarz quadrangle of the Heun equation into D Schwarz triangles of the hypergeometric equation.

In the context of hyperbolic geometry, the possible subdivisions of curvilinear quadrangles (or triangles) into curvilinear triangles have been classified by Felixson [7]; they are called *Coxeter decompositions*. The triangles have angles $\pi\alpha, \pi\beta, \pi\gamma$ satisfying $\alpha + \beta + \gamma < 1$. The Coxeter decompositions with a free (angle) parameter

are depicted in Figures 10, 11, 14 in [7]. The subdivisions of Schwarz quadrangles into Schwarz triangles induced by our Gauss-to-Heun transformations defined over \mathbb{R} have the same shape. In Tables 1–3,

- the notation F_k refers to the k th subdivision picture in [7, Figure 14]; these subdivisions are applicable to Gauss-to-Heun pull-backs of the type $(2, 3)$;
- F'_k similarly refers to [7, Figure 11]; these subdivisions are applicable the pull-backs of the types (2) , $(2, 4)$, $(2, 5)$, $(2, 6)$;
- F''_k similarly refers to [7, Figure 10]; these subdivisions are applicable the pull-backs of the type (3) or $(3, 3)$.

Figure 2(a) depicts the Coxeter decomposition F'_{13} of a quadrangle with the angles $\pi\alpha, \pi\alpha, 2\pi\alpha, 2\pi\alpha$ into 6 hyperbolic triangles with the angles $\pi/2, \pi/6, \pi\alpha$. It gives a decomposition of a Schwarz quadrangle for $HE(\alpha, \alpha, 2\alpha, 2\alpha)$ into Schwarz triangles for $E(1/2, 1/6, \alpha)$ induced by the type $(2, 6)$ transformation with the covering H_{39} . The Schwarz reflection principle is applied to a few edges intersecting at a common vertex. The decompositions $3 \cdot 2$ and $2_H \cdot 3$ are clearly visible in the Coxeter decomposition. Consequently, the picture also illustrates the decomposition F''_2 of the same quadrangle into 3 triangles with the angles $\pi/3, \pi\alpha, \pi\alpha$, and the decomposition F_2 of a quadrangle with the angles $\pi/2, \pi/2, \alpha, 2\alpha$. Both decompositions are induced by the cubic covering H_{34} . The factor 2_H represents a Schwarz reflection between two smaller quadrangles.

Figure 2(b) is not a quadrangle, of course. But it contains two Coxeter decompositions for Gauss-to-Heun transformations of the type $(3, 3)$. If we remove the upper black triangle, we get the decomposition F''_3 of a quadrangle with the angles $\pi/3, \pi/3, 2\pi\alpha, 2\pi\alpha$. If the left white triangle is removed, the decomposition F''_4 of a quadrangle with the angles $\pi/3, \pi/3, \pi\alpha, 3\pi\alpha$ is obtained. The coverings are H_{47} and H_{46} , respectively.

Similarly, Figure 2(c) includes all Coxeter decompositions for the Gauss-to-Heun transformations of the types $(2, 4)$ and $(2, 5)$. Here we identify the quadrangles (and the corresponding Belyi coverings) for the Coxeter decompositions F'_6 to F'_{12} , respectively:

$$\begin{aligned} ABCF (H_{25}), \quad ABFH (H_{41}), \quad ABDF (H_{20}), \quad BDFH (H_{40}), \\ ABML (H_{35}), \quad ABCL (H_{29}), \quad OCEG (H_{45}). \end{aligned}$$

The quadrangles (and coverings) for the Coxeter decompositions F'_{14} and F'_{15} are $KCFH (H_{43})$ and $OCEH (H_{24})$, respectively.

Finally, Figure 2(c) includes all Coxeter decompositions for the Gauss-to-Heun transformations specifically of the type $(2, 3)$. They are numbered from F_5 to F_{27} by [7, Figure 10]. Here are their quadrangles (and coverings), respectively:

$$\begin{aligned} AOEX (H_{30}), \quad AXEZ (H_{28}), \quad AXEY (H_{27}), \quad AFOY (H_{29}), \quad AFOQ (H_{24}), \\ AOEP (H_{25}), \quad AXEQ (H_{22}), \quad AXER (H_{23}), \quad AFEO (H_{17}), \quad AFOB (H_{15}), \\ APER (H_{20}), \quad APEQ (H_{19}), \quad AOEG (H_{16}), \quad AXED (H_{13}), \quad APED (H_9), \\ AXEB (H_{12}), \quad APEB (H_8), \quad ACEF (H_5), \quad ABEG (H_2), \quad ADEG (H_4), \\ AXEC (H_{14}), \quad APEC (H_{10}), \quad ACEG (H_3). \end{aligned}$$

In total, there are $(27 - 4) + (15 - 5) + (4 - 2) = 35$ subdivisions F_k , F'_k , F''_k representing Gauss-to-Heun transformations with exactly one parameter.

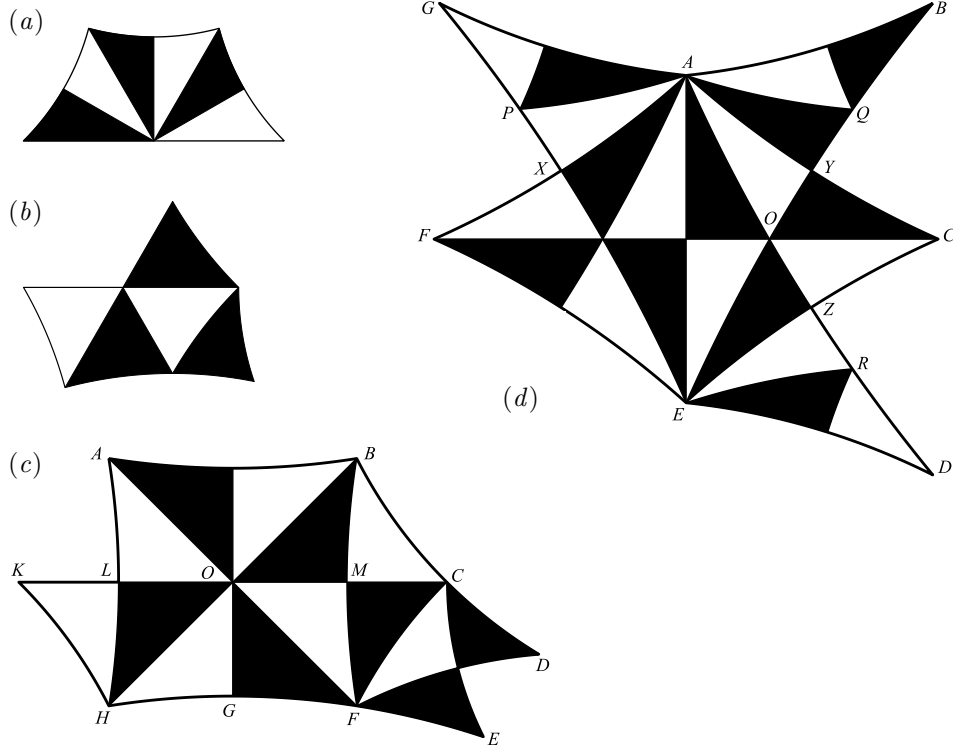


FIGURE 2. Coxeter decompositions for the parametric Gauss-to-Heun transformations defined over \mathbb{R}

The subdivisions for the Gauss-to-Heun transformations with 2 or 3 parameters are the following:

- The Coxeter decomposition for quadratic transformation (2.8) is represented by a single Schwarz reflection. It can be discerned in many places in Figure 2, for example as the quadrangle $OYCZ$ in picture (d). It appears several times in Felikson's figures, in particular as $F_1 = F'_1 = F''_1$.
- There are two degree 3 decompositions $F_2 = F'_2$ and F''_2 . They are both represented by the covering H_{34} , as we mentioned discussing picture (a). The other cubic transformation (with the covering H_{33}) is not defined over \mathbb{R} in the normalized form [30, §4.4.4] but over $\mathbb{Q}(\omega)$, hence there is no Coxeter decomposition for it.
- There are three degree 4 decompositions, $F_3 = F'_3$, F''_3 and $F_4 = F'_4$. They can be discerned, for example, as the following quadrangles in picture (c), respectively: $OBCF$ (H_{31}), $OABC$ (H_{47}), $OCEF$ (H_{35}).

Whether a Gauss-to-Heun transformation is realized by a Coxeter decomposition, is determined by a close inspection in Step 4 of §3. A necessary and sufficient condition is that the Belyi covering has to be defined over \mathbb{R} after a normalization (by Möbius transformations) that locates 3 of the 4 singular points of Heun's equation as $x = 0$, $x = 1$, $x = \infty$. In particular, the fourth singular point $x = t$ has to be real, though this is not a sufficient condition. For example, a proper

normalization of H_{48} for the type (4, 4) transformation is $8ix(x^2 - 1)/(x + i)^4$. This gives $t = -1$, but the covering is not defined over \mathbb{R} . There is one other example of this type: a proper normalization of H_{28} for a type (3, 3) pull-back is $3(1 + 2\omega)x^2(x^2 - 1)/(x^2 + \omega)^3$. On the other hand, a proper normalization of the same H_{28} for a type (2, 3) pull-back is defined over $\mathbb{Q}(\sqrt{3})$, giving the Coxeter decomposition F_6 . There are two different Coxeter decompositions for each of the following coverings: H_{20} , H_{24} , H_{25} , H_{29} , H_{34} , H_{35} , H_{47} . Comparison of our classification and Felikson's list [7] provides a useful mutual confirmation.

The considered Coxeter decompositions are *parametrized*, in that one or more of the triangular vertex angles are free to vary. For somewhat larger real values of the free parameter(s), the Coxeter decompositions are transfigured to spherical geometry of the Riemann sphere (if angles larger than π are allowed), as subdivisions of spherical quadrangles into spherical triangles with the angles satisfying $\alpha\pi + \beta\pi + \gamma\pi > \pi$. Most of the Coxeter decompositions can be transfigured to the plain Euclidean geometry (where $\alpha\pi + \beta\pi + \gamma\pi = \pi$) as well. The exceptions are F_{14} , F_{16} , F_{20} , F_{27} , F_9' , F_4'' , for which the quadrangles degenerates to flat triangles.

Broughton et al. [5] classify similar geometric objects: *divisible tilings* of the hyperbolic plane. Compared with Felikson's pictures, divisible tilings form a proper subset of Coxeter decompositions. The condition for a Coxeter decomposition to be a divisible tiling is that the quadrangle angles be equal to π/k , with k an integer. In general Coxeter decompositions, *rational* multiples of π are also allowed. The one-parameter divisible tilings relevant here are depicted in [5, Table 6.6]. There are 34 of them; the first 6 correspond to Gauss-to-Heun transformations with 2 or 3 parameters. Divisible tilings are indicated in Tables 1–3 by the notation F_7^*, \dots, F_{34}^* , where the subscripts refer to the numbering in [5, Table 6.6]. There are $35 - (34 - 6) = 7$ relevant Coxeter decompositions with one parameter that are not divisible tilings; they all have the angle $2\pi/3$.

4.4. Composite transformations. The composite Gauss-to-Heun transformations can be inductively deduced from a smaller set of pull-back transformations among hypergeometric and Heun functions. Due to the associativity of the composition operation, one can always decompose a Gauss-to-Heun transformation as a product of the following:

- A possibly composite Gauss-to-Gauss transformation with a free parameter, excluding Möbius transformations and from $E(1, \alpha, \alpha)$ or $E(1/2, 1/2, \alpha)$ for the purposes of this article. This could be the quadratic transformation (2.6) and one of 6 classical transformations (of degrees 3, 4 and 6) worked out by Goursat [8] and listed in [29, Table 1].
- An *indecomposable* Gauss-to-Heun transformation with at least one free parameter. This could be the quadratic transformation (2.8); one of 4 other indecomposable transformations of Table 1; or an indecomposable transformation of Table 2 or 3 of degree at most 6, possibly fitting a Gauss-to-Gauss or a Heun-to-Heun transformation.
- A Heun-to-Heun transformation with at least one free parameter. This could be the quadratic transformation (2.7), or the degree 4 composite transformation

$$(4.2) \quad HE(1/2, 1/2, 1/2, \alpha) \xleftarrow{2H} HE(1/2, 1/2, \alpha, \alpha) \xleftarrow{2H} HE(\alpha, \alpha, \alpha, \alpha),$$

realized by the covering H_{31} . See [30, §4.3] for an overview.



Figure 3 graphically depicts all possible compositions of the preceding three types (Gauss-to-Gauss, etc.) The two longest boxes, centrally placed, represent quadratic transformations (based on the double covering H_{32}). The following objects and information are included in the figure.

There are 7 boxes with double edges on the left and the right sides, representing the classical Gauss-to-Gauss transformations. The quadratic transformation appears as the long box in the lower part; two indecomposable transformations (of degree 3 or 4) appear as boxes in the central part; and the remaining four classical transformations (of degrees 3, 4 and 6) are represented in the upper part. Of the latter, only the cubic transformation is indecomposable. The transformation appearing near the upper-right corner can be decomposed in two different ways; its covering does not occur in Tables 1–3, hence it is not identified by an H number. These 7 boxes will be called $E \rightarrow E$ boxes.

The 10 other boxes represent indecomposable Gauss-to-Heun transformations. The quadratic transformation (2.6) is represented by the long box in the upper part; the four indecomposable transformations of Table 1 appear in the central part. The three isolated boxes near the lower right corner represent the indecomposable transformations of Table 3, to each of which the quadratic Heun-to-Heun transformation (2.7) can be applied. The other two lowest boxes represent transformations in Table 3 that can be composed with a specialization of the quadratic $E \rightarrow E$ transformation. These 10 boxes will be called $E \rightarrow HE$ boxes.

The vertical lines connect $E \rightarrow E$ and $E \rightarrow HE$ boxes whose transformations can be composed (perhaps after a specialization of parameters). The composed coverings are labeled by H numbers on the left side of each vertical line. Relevant specializations of the quadratic $E \rightarrow E$ transformation are given as well. Note that the specializations $p = \frac{1}{2}$ and $q = \frac{1}{2}$ of the quadratic $E \rightarrow E$ transformation are not given, because (as stated above) the dihedral family is not considered here. The number of possible compositions between an $E \rightarrow E$ box and an $E \rightarrow HE$ box depends on the number of ways to identify (without degeneracy) the exponent differences of the intermediate hypergeometric equation. It is instructive to compare compositions of the quadratic $E \rightarrow HE$ transformation with the two hypergeometric transformations coming from the coverings H_{47} and H_{34} . Compositions of the quadratic $E \rightarrow E$ and $E \rightarrow HE$ transformations occur as the composite quartic coverings H_{35} , H_{31} in Table 1.

The \Rightarrow symbols outside the boxes indicate application of the quadratic Heun-to-Heun transformation (2.7). If this transformation can be applied after an indecomposable Gauss-to-Heun transformation, the relevant parameter specializations and composite coverings are indicated to the right (or near the lower right corner) of the respective box. If (2.7) can be applied after a composite Gauss-to-Heun transformation, this is indicated by the \Rightarrow symbol to the right of the H number of the composite covering (and to the right of the respective vertical line).

Some boxes of the same kind touch each other, but that does not have a particular meaning. The box for the quadratic $E \rightarrow HE$ transformation (2.8) is connected to all $E \rightarrow E$ boxes, since this transformation can always be applied without restrictions on the exponent differences. The box for the quadratic $E \rightarrow E$ transformation (2.6) is connected to all $E \rightarrow HE$ boxes, except for the isolated three.

To show completeness of Figure 3, one must:

- Consider all transformations of Tables 2, 3 to which the quadratic Heun-to-Heun transformation (2.7) can be applied; and after computing and examining the resulting compositions, keep only indecomposable transformations.
- Check the classical $E \rightarrow E$ transformations of [29, Table 1] and the quadratic $E \rightarrow E$ transformations, to determine whether the pulled-back hypergeometric equation can ever have exponent differences of the form $1/k, 1/\ell$, consistent with (3.7); if so, composition with a one-parameter $E \rightarrow HE$ transformation of Table 2 or 3 may be possible.
- If a pair of $E \rightarrow E$ and $E \rightarrow HE$ boxes is not connected by a vertical line, check that the respective transformations cannot be composed.
- Check completeness of coverings for each vertical line.
- Check possible compositions with the Heun-to-Heun transformations of degrees 2 and 4.

The information of Figure 3 is given in the rightmost columns of Tables 1, 2, 3. The compositions are spelled out more explicitly in [30, Appendix B]. A multiple occurrence of a covering in Figure 3 means either that it can be decomposed in more than one way (as for $H_3, H_5, H_6, H_{19}, H_{31}, H_{39}, H_{41}$); or that it appears in more than one composition (as for $H_{25}, H_{28}, H_{35}, H_{38}$); or both (as for H_{20}).

The following cases are worth attention. Firstly, there are three ways to decompose the quartic covering H_{31} in Table 1, on account of its special branching pattern $2+2=2+2=2+2$:

$$(4.3) \quad H_{31}: \begin{cases} E(1/2, \alpha, \beta) \xleftarrow{2} E(\alpha, \alpha, 2\beta) \xleftarrow{2} HE(2\alpha, 2\alpha, 2\beta, 2\beta), \\ E(1/2, \alpha, \beta) \xleftarrow{2} E(2\alpha, \beta, \beta) \xleftarrow{2} HE(2\alpha, 2\alpha, 2\beta, 2\beta), \\ E(1/2, \alpha, \beta) \xleftarrow{2} HE(1/2, 1/2, 2\alpha, 2\beta) \xleftarrow{2H} HE(2\alpha, 2\alpha, 2\beta, 2\beta). \end{cases}$$

This is indicated by the 2×2 in the rightmost column. The covering H_{31} occurs as a part of the larger compositions H_5 and H_{41} ; see their composition lattices in [30, (B.5), (B.4)]. Besides, the covering H_{31} induces the degree 4 Heun-to-Heun transformation (4.2).

The transformation $E(1/2, 1/4, \alpha) \xleftarrow{4} HE(1/2, 1/2, 2\alpha, 2\alpha)$ is induced by two distinct coverings: H_{31} and H_{35} . Induced by H_{31} , this transformation is the $\beta = 1/4$ specialization of (4.3); induced by H_{35} , this transformation is a new one suggested by the branching pattern given in Table 3. Both transformations have the factorization

$$(4.4) \quad E(1/2, 1/4, \alpha) \xleftarrow{2} HE(1/2, 1/2, 1/2, 2\alpha) \xleftarrow{2H} HE(1/2, 1/2, 2\alpha, 2\alpha),$$

but they have different sets of t parameters. Both H_{31} and H_{35} appear as parts of the degree 8 composite transformation H_{41} .

5. EXISTENCE AND UNIQUENESS OF COVERINGS

This section presents an elegant way to conclude that there are no Belyi coverings with some branching patterns. The idea is to deduce a pull-back transformation of Fuchsian equations that is not possible, because it would relate an equation with finite monodromy to an equation with infinite monodromy group, or the pulled-back equation would not exist. We apply this idea to all cases of non-existent coverings of Tables 1, 2, 3. Moreover, in §5.3 this approach is applied to most cases of non-existent coverings in the Miranda–Persson list [18] of K3 elliptic surfaces.

As an immediate example, consider the non-existent covering of degree 4 in Table 1. If it would exist, the specialization $\alpha = 1/2$ would give a pull-back from $E(1/2, 1/2, \beta)$ to a Fuchsian equation with two singularities and (generally) non-equal exponent differences $\beta, 3\beta$ at them, contradicting part (ii) of Lemma 5.1 below. Or one can further specialize $\beta = 1$ or $\beta = 1/3$ and get a contradiction with part (i) of the same lemma. In §5.1 we prove several assertions from which we make non-existence conclusions. Table 5 outlines the non-existence proofs. In §5.4, we seek to show uniqueness (up to Möbius transformations) of the Belyi coverings with the encountered branching patterns, by considering implied pull-backs between Fuchsian equations with finite monodromy groups.

5.1. Principal lemmas. The easiest way to conclude non-existence of a Belyi covering with a certain branching pattern is to deduce a pull-back transformation to a non-existent Fuchsian equation. Here are two basic situations.

- Lemma 5.1.** (a) *There is no Fuchsian equation on \mathbb{P}^1 that has exactly one relevant singular point.*
 (b) *If a Fuchsian equation on \mathbb{P}^1 has exactly 2 singular points, their exponent differences are equal.*

Proof. If a Fuchsian equation has just one relevant singularity, we can move it to infinity and make all points in \mathbb{C} ordinary. The differential equation then has the form $y'' + Py' + Qy = 0$, where P, Q are polynomials (in the differentiation variable x). If $P = Q = 0$, then the local exponents at the infinity are $0, -1$, thus $x = \infty$ will be an irrelevant singularity. Otherwise $x = \infty$ is an irregular singularity, and the equation will not be Fuchsian.

If a Fuchsian equation has 2 singularities, we can assume them to be $x = 0$, $x = \infty$. The Liouville normal form of the equation is then $x^2 y'' = cy$ with $c \in \mathbb{C}$. The exponent differences of this equation equal $\sqrt{1 + 4c}$ at both singular points. \square

Another type of non-existent transformation is a pull-back of a hypergeometric equation with finite monodromy to a hypergeometric equation with infinite monodromy. (A Fuchsian equation has finite monodromy if and only if its solution space has a basis consisting of algebraic functions.) The following lemma characterizes some hypergeometric equations with finite (or infinite) monodromy groups.

Lemma 5.2. *Consider a hypergeometric equation $E = E(\alpha, \beta, \gamma)$ on \mathbb{P}^1 .*

- (a) *Suppose that α, β, γ are rational numbers, each having denominator 3. Then the monodromy of E will be finite if and only if the sum of the numerators of α, β, γ is even.*
- (b) *If α is a half-odd-integer, and β, γ are rational numbers, each having denominator 4, then the monodromy of E is not finite.*
- (c) *Suppose that α, β, γ are integers. Then the monodromy of E will be trivial if and only if the sum $\alpha + \beta + \gamma$ is odd, and the triangle inequalities $\gamma < \alpha + \beta$, $\beta < \alpha + \gamma$, $\alpha < \beta + \gamma$ are satisfied; otherwise the monodromy is not finite.*
- (d) *Suppose that α is an integer while β, γ are half-odd-integers. The set $\{|\beta - \gamma|, \beta + \gamma\}$ contains two integers of different parity; let k be the integer in this set such that $k + \alpha$ is odd. Then the monodromy group of E will be isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if and only if $k < \alpha$; otherwise the monodromy will not be finite.*

Proof. We use the Schwarz classification of hypergeometric equations with finite monodromy for the first two statements; see [23] or [6, § 2.7.2]. The only possible projective monodromy in statement (a) is the tetrahedral group A_4 . There are two Schwarz types (II and III) for this group: the hypergeometric equation must be contiguous either to $E(1/2, 1/3, 1/3)$ or to $E(1/3, 1/3, 2/3)$. We must have the latter Schwarz type III. Contiguous relations shift the exponent differences by integers whose sum is even. That does not change the parity of the numerator sum (of the three integers divided by 3), even if an exponent difference is multiplied by -1 .

We do not find the denominator pattern of the statement (b) in the Schwarz list. In particular, the two Schwarz types (IV and V) for the octahedral group S_4 are contiguous to $E(1/2, 1/3, 1/4)$ or $E(2/3, 1/4, 1/4)$.

For the claim (c), a representative solution of the generic hypergeometric equation with trivial monodromy is ${}_2F_1(-n, \ell+1; -n-m \mid z)$, with n, m, ℓ non-negative integers; see [27, Theorem 2.4(5)]. Up to a permutation, one has that $\alpha = n+m+1$, $\beta = n+\ell+1$, $\gamma = m+\ell+1$; that is

$$(5.1) \quad n = \frac{\alpha + \beta - \gamma - 1}{2}, \quad m = \frac{\alpha + \gamma - \beta - 1}{2}, \quad \ell = \frac{\beta + \gamma - \alpha - 1}{2}.$$

If one of these three numbers is a negative integer, the singular point with the largest exponent difference is logarithmic [27, § 9]. If each of the above three numbers is a half-odd-integer, all three singular points are logarithmic [27, § 5].

The assertion (d) is a reformulation of [28, Theorem 5.1], stated in the context of hypergeometric equations with either logarithmic solutions or the $\mathbb{Z}/2\mathbb{Z}$ monodromy group. \square

Existence (and uniqueness) of coverings with a given branching pattern can also be decided on the basis of transformations of some hypergeometric equations with infinite monodromies. The following lemma implies that there are no transformations of $E(1/2, 1/4, 1/4)$ into itself of degrees 6, 12, 14, 21, 22, 24, or generally, of degrees $\equiv 3 \pmod{4}$, even if suitable branching patterns of these degrees exist. Similarly, there are no transformations of $E(1/2, 1/3, 1/6)$ or $E(1/3, 1/3, 1/3)$ into themselves of degrees 6, 10, 15, 18, 22, 24, or generally, of degrees $\equiv 2 \pmod{3}$. This lemma elucidates the non-uniqueness of H_{44} and H_{21} .

- Lemma 5.3.** (a) *Up to Möbius transformations, the number of degree- D pull-back coverings of $E(1/2, 1/4, 1/4)$ into itself is equal to the number of integer solutions (a, b) with $a \geq 0$, $b > 0$, of the equation $D = a^2 + b^2$.*
- (b) *Up to Möbius transformations, the number of degree- D pull-back coverings of $E(1/2, 1/3, 1/6)$ or $E(1/3, 1/3, 1/3)$ into itself is equal to the number of integer solutions (a, b) with $a \geq 0$, $b > a$, of the equation $D = a^2 - ab + b^2$.*

Proof. According to [29, § 8], the transformations of $E(1/2, 1/4, 1/4)$ into itself correspond to isogenies of the $j = 1728$ elliptic curve $y^2 = x^3 - x$. The ring of isogenies is isomorphic to the ring $\mathbb{Z}[i]$ of Gaussian integers, and the degree of a pull-back is equal to the norm $a^2 + b^2$ of the corresponding $a + bi$. In particular, the trivial and fractional-linear transformations correspond to the units $\pm 1, \pm i$. Therefore one must count $a + bi \in \mathbb{Z}[i]$ such that $|a + bi|^2 = D$ and $\arg(a + bi) \in [0, \pi/2)$.

Similarly [29, § 8], the transformations of $E(1/2, 1/3, 1/6)$ or $E(1/3, 1/3, 1/3)$ into itself correspond to isogenies of the $j = 0$ elliptic curves $y^2 = x^3 - 1$ or $x^3 + y^3 = 1$. The ring of isogenies is isomorphic to the ring of Eisenstein integers $\mathbb{Z}[\omega]$. The degree of a pull-back is equal to the norm $a^2 - ab + b^2$ of the corresponding $a + b\omega$. Trivial

Nonexistent covering	Deg. D	Branching pattern above singular points	Lemma	Exponent differences	
				hypergeom.	pulled-back
N_1	12	$[2]_6 = [3]_4 = 7+3+1+1$	5.2(a)	$1/2, 1/3, 1/3$	$7/3, 1/3, 1/3$
N_2		$[2]_6 = [3]_4 = 7+2+2+1$	5.1(b)	$1/2, 1/3, 1/2$	$1/2, 7/2$
N_3		$[2]_6 = [3]_4 = 6+4+1+1$	5.2(b)	$1/2, 1/3, 1/4$	$3/2, 1/4, 1/4$
N_4		$[2]_6 = [3]_4 = 6+2+2+2$	5.1(a)	$1/2, 1/3, 1/2$	3
N_5		$[2]_6 = [3]_4 = 5+4+2+1$	5.2(d)	$1/2, 1/3, 1/2$	$2, 1/2, 5/2$
N_6		$[2]_6 = [3]_4 = 5+3+3+1$	5.1(b)	$1/2, 1/3, 1/3$	$1/3, 5/3$
N_7		$[2]_6 = [3]_4 = 5+3+2+2$	5.1(b)	$1/2, 1/3, 1/2$	$3/2, 5/2$
N_8		$[2]_6 = [3]_4 = 4+4+3+1$	5.1(b)	$1/2, 1/3, 1/4$	$1/4, 3/4$
N_9		$[2]_6 = [3]_4 = 4+3+3+2$	5.1(b)	$1/2, 1/3, 1/3$	$2/3, 4/3$
N_{10}	10	$[2]_5 = [3]_3 + 1 = 6+3+1$	5.3(b)	$1/2, 1/3, 1/6$	$1/2, 1/3, 1/6$
N_{11}		$[2]_5 = [3]_3 + 1 = 6+2+2$	5.1(b)	$1/2, 1/3, 1/2$	$1/3, 3$
N_{12}		$[2]_5 = [3]_3 + 1 = 4+4+2$	5.1(b)	$1/2, 1/3, 1/4$	$1/3, 1/2$
N_{13}		$[2]_5 = [3]_3 + 1 = 4+3+3$	5.1(b)	$1/2, 1/3, 1/3$	$1/3, 4/3$
N_{14}	9	$[2]_4 + 1 = [3]_3 = 5+2+2$	5.1(b)	$1/2, 1/3, 1/2$	$1/2, 5/2$
N_{15}		$[2]_4 + 1 = [3]_3 = 4+4+1$	5.1(b)	$1/2, 1/3, 1/4$	$1/2, 1/4$
N_{16}		$[2]_4 + 1 = [3]_3 = 3+3+3$	5.1(a)	$1/2, 1/3, 1/3$	$1/2$
N_{17}	8	$[2]_4 = [3]_2 + 2 = 4+3+1$	5.2(a)	$1/2, 1/3, 1/3$	$1/3, 2/3, 4/3$
N_{18}		$[2]_4 = [3]_2 + 2 = 4+2+2$	5.1(b)	$1/2, 1/3, 1/2$	$2, 2/3$
N_{19}		$[2]_4 = [3]_2 + 1 + 1 = 5+3$	5.2(a)	$1/2, 1/3, 1/3$	$1/3, 1/3, 5/3$
N_{20}	6	$[2]_3 = [3]_1 + 2 + 1 = 3+3$	5.1(b)	$1/2, 1/3, 1/3$	$1/3, 2/3$
N_{21}		$[2]_4 = [4]_2 = 5+1+1+1$	5.2(c)	$1/2, 1/2, 1$	$2, 2, 5$
N_{22}	6	$[2]_4 = [4]_2 = 3+2+2+1$	5.1(b)	$1/2, 1/4, 1/2$	$1/2, 3/2$
N_{23}		$[2]_3 = [4]_1 + 2 = 4+1+1$	5.1(b)	$1/2, 1/2, 1$	$4, 2$
N_{24}	6	$[2]_3 = [4]_1 + 2 = 2+2+2$	5.1(a)	$1/2, 1/4, 1/2$	$1/2$
N_{25}		$[2]_3 = [5]_1 + 1 = 2+2+2$	5.1(a)	$1/2, 1/5, 1/2$	$1/5$
N_{26}	6	$[3]_2 = [3]_2 = 3+1+1+1$	5.1(a)	$1/3, 1/3, 1$	3
N_{27}	4	$[2]_2 = 3+1 = 2+2$	5.1(a)	$1/2, 1/3, 1/2$	$1/3$

TABLE 5. Unrealizable branching patterns, with a proof indication

or Möbius transformations correspond to the units $\pm 1, \pm \omega, \pm(\omega + 1)$. Therefore one must count $a + b\omega \in \mathbb{Z}[\omega]$ such that $|a + b\omega|^2 = D$ and $\arg(a + b\omega) \in [0, \pi/3)$. \square

5.2. Nonexistence of coverings. Tables 2, 3 have 27 entries with nonexistent Belyi coverings. One branching pattern appears twice among the type $(2, 4)$ candidates, hence the two tables actually have 26 different branching patterns with no covering. They are labeled N_1, \dots, N_{26} . The repeating branching pattern is labelled N_{23} . Nonexistence is in each case an immediate consequence of some lemma in §5.1. Mostly by specialization of the free parameter, one either derives a pull-back from a hypergeometric equation to a nonexistent Fuchsian equation, or a pull-back of a hypergeometric equation with finite monodromy to a hypergeometric equation with infinite monodromy, or a nonexistent pull-back of $E(1/2, 1/3, 1/6)$ into itself. The unrealizable branching patterns and the applicable lemmas are listed in Table 5.

The non-existent covering of Table 1 is given the last number N_{27} . Its non-existence was already demonstrated at the beginning of this section.

Only for N_{21} and N_{23} the used implied transformation is not a specialization of a respective Gauss-to-Heun pull-back of the classification in §3. To prove N_{21} by

the specialization $\alpha = 1/5$, one would need to inspect the 10 icosahedral Schwarz types in [6, §2.7.2]. The case N_{23} can be proved using the specialization $\alpha = 1/4$ of either of the two candidate transformations in Table 3, by invoking Lemma 5.3(b). Note that to use a hypergeometric equation with only two relevant singularities, one must ensure that it is of the form $E(1, \alpha, \alpha)$. In particular, Lemma 5.1(b) does not apply to the branching covering $[2]_6=[3]_4=9+1+1+1$ and its pull-backs from $E(1/2, 1/3, 1)$, because logarithmic singularities rather than ordinary points appear. And indeed, the covering H_1 exists.

5.3. The Miranda-Persson classification. The lemmas of §5.1 can be applied to the problem of the existence of Belyi maps that would yield semi-stable elliptic fibrations of K3 surfaces with 6 singular fibers, sorted out by Miranda, Person [18] and Beukers, Montanus [4]. The degree of the relevant Belyi maps is 24, and their branching patterns have the form $[2]_{12}=[3]_8=P$, where $P = a+b+c+d+e+f$ is a partition of 24 with exactly 6 parts. There are 199 of these branching patterns in total. Miranda and Persson [18] proved that Belyi coverings (and elliptic K3 surfaces) exist in 112 cases, and do not exist in the remaining 87 cases. Beukers and Montanus [4] computed all¹ those Belyi maps and checked non-existence for the 87 partitions.

The non-existence proof in [18] broadly relies on two techniques. First, Miranda and Persson widen the space of considered branching patterns to include partitions P with more than six parts² and conclude non-existence of coverings for a partition $a_1 + \dots + a_s$ from non-existence for a partition $a_1 + \dots + a_{s-1} + a'_s + a''_s$ with $a_s = a'_s + a''_s$, using [18, Lemma (2.4)]. Secondly, they get contradicting conclusions about the torsion of the assumed elliptic surfaces in several non-existing cases. In [4], non-existence is concluded either by using a sum over the characters of S_{24} that counts coverings (not necessarily connected, with some rational weights) with a given branching pattern, or by direct computation. Let Σ denote the counting character sum just mentioned, given in [4, Theorem 3.2]. The large table in [4] does not list the 47 partitions (out of the total 87) for which $\Sigma = 0$.

Here we show that most of the non-existent cases in the Miranda-Persson list can be deduced using the methods of §5.1. Here are 22 partitions out of the 40 ones with $\Sigma \neq 0$ for which the non-existence can be proved by using Lemmas 5.1, 5.2, 5.3 directly:

$$14+[2]_5, 9+[3]_5, 15+[2]_4+1, 13+3+[2]_4, 12+4+[2]_4, 11+5+[2]_4, 10+6+[2]_4,$$

¹As pointed out in the AMS MathSciNet review by David P. Roberts, the table in [4] omits one Belyi covering for the partition $10+6+4+2+1+1$. Our computation confirms existence of two (rather than one) Belyi coverings for this partition:

$$\frac{(144x^8 + 384x^7 + 1120x^6 - 784x^3 + 756x^2 - 240x + 25)^3}{108x^6(14x-5)^4(4x-1)^2(9x^2+24x+70)},$$

$$\frac{(144x^8 - 1536x^7 + 5248x^6 - 5568x^5 - 720x^4 + 512x^3 + 192x^2 + 24x + 1)^3}{108(8x+1)^6x^4(x-3)^2(9x^2-42x-5)}.$$

The second covering is missing in the Beukers-Montanus list. In total, there are 59 branching patterns (among the 112 indicated by Miranda and Persson) with a unique Belyi map up to Möbius transformations; 125 Galois orbits of the Belyi maps, of size at most 4; and 191 different Belyi maps or *dessins d'enfant*.

²Therefore coverings with more than 3 branching fibers are allowed. Instead of the coverings, permutation representations of their monodromy are considered in [18].

$$11+[3]_4+1, 10+[3]_4+2, 8+4+[3]_4, 13+4+[2]_3+1, 11+6+[2]_3+1, 11+4+3+[2]_3, \\ 10+4+4+[2]_3, 9+8+[2]_3+1, 9+6+3+[2]_3, 8+7+3+[2]_3, 8+5+5+[2]_3, 7+7+4+[2]_3, \\ 10+[4]_3+1+1, 6+[4]_3+3+3, [6]_3+3+2+1.$$

The choice of the starting $E(1/2, 1/3, 1/k)$ that yields a non-existent covering is indicated by the $[k]_n$ notation. Next, here are 22 partitions out of the 47 ones with $\Sigma = 0$ to which our lemmas apply directly:

$$9+7+[2]_4, 7+5+[3]_4, 7+[4]_4+1, 6+[4]_4+2, 5+[4]_4+3, [5]_4+3+1, \\ 9+5+4+[2]_3, 7+6+5+[2]_3, 13+[3]_3+1+1, 11+[3]_3+2+2, 10+4+[3]_3+1, \\ 8+6+[3]_3+1, 8+5+[3]_3+2, 7+7+[3]_3+1, 7+6+[3]_3+2, 7+4+4+[3]_3, 6+5+4+[3]_3, \\ 5+5+5+[3]_3, 9+[4]_3+2+1, 6+5+[4]_3+1, 7+[4]_3+3+2, 5+5+[4]_3+2.$$

Additionally, the four cases $7+[5]_3+1+1$, $6+[5]_3+2+1$, $[5]_3+4+4+1$, $[5]_3+4+3+2$ with $\Sigma = 0$ are concluded by inspecting the icosahedral hypergeometric equations in the Schwarz table [6, § 2.7.2]. In total, this shows 48 out of the 87 cases.

More cases of non-existence can be deduced from implied pull-backs to Fuchsian equations with 3 non-apparent singularities and a few apparent singularities. These equations are gauge “contiguous” to target hypergeometric equations (with infinite or infinite monodromy) as the local exponent differences differ at all points by integers. The total shift of the exponent differences, including those from the difference 1 for ordinary points of hypergeometric equations, must be an even integer. In this way, non-existence for the following 7 partitions with $\Sigma \neq 0$ can be shown:

$$10+6+[3]_2+1+1, 9+9+[3]_1+1+1+1, 8+6+[3]_2+2+2, 7+6+6+[3]_1+1+1, \\ 7+6+4+[3]_2+1, 6+5+5+[3]_2+2, 8+6+[4]_2+1+1.$$

In each case, the apparent singularities are represented by the branching orders that are integer multiples of the bracketed numbers. And here are 7 partitions with $\Sigma = 0$ that can be handled in the same way:

$$9+7+[3]_2+1+1, 9+5+[3]_2+2+2, 9+4+4+[3]_2+1, 6+6+5+[3]_1+2+2, \\ 6+6+4+4+[3]_1+1, 8+5+[4]_2+2+1, 8+[4]_2+3+3+2.$$

Besides, a pull-back from $E(1/2, 1/3, 1/3)$ can be applied to show the non-existence for $9+6+6+1+1+1$, with $\Sigma \neq 0$. It is trickier to combine parts (c), (d) of Lemma 5.2 with gauge shifts.

Of the remaining $87 - 48 - 7 - 7 - 1 = 24$ partitions, the following 6 (with $\Sigma \neq 0$) and 11 (with $\Sigma = 0$) partitions could be handled with a full knowledge of Heun equations with finite monodromy (that are not classified yet):

$$10+8+[2]_2+1+1, 13+[4]_2+1+1+1, 11+[4]_2+2+2+1, 9+[4]_2+3+2+2, \\ 9+[5]_2+3+1+1, 8+[5]_2+4+1+1; \\ 9+6+4+[2]_2+1, 8+8+3+[2]_2+1, 7+7+5+[2]_2+1, 7+6+4+3+[2]_2, \\ 10+5+[3]_2+2+1, 7+5+5+[3]_2+1, 9+5+[4]_2+1+1, 7+5+[4]_2+2+2, \\ 8+[5]_2+3+2+1, 6+[5]_2+4+3+1, 6+[5]_2+4+2+2.$$

Besides, a pull-back from $E(1/2, 1/3, 1/4)$ could be then applied to two partitions with $\Sigma = 0$: $12+8+1+1+1+1$, $8+8+5+1+1+1$. Other 3 partitions (with $\Sigma \neq 0$)

$$12+5+[4]_1+1+1+1, 10+[5]_1+4+3+1+1, 9+8+[4]_1+1+1+1,$$

could be decided by Fuchsian equations with 4+1 singularities (i.e., 4 non-apparent and 1 apparent). There remain only two partitions: $7+7+6+2+1+1$ with $\Sigma \neq 0$, and $7+7+4+3+2+1$ with $\Sigma = 0$. Their non-existence might be decided by using implied pull-backs from $E(1/2, 1/3, 1/2)$ to Fuchsian equations with 4+1 singularities and the monodromy group D_2 or $\mathbb{Z}/2\mathbb{Z}$.

5.4. Uniqueness of coverings. Uniqueness of Gauss-to-Heun transformations (and of their coverings) with a plausible branching pattern can be concluded from uniqueness of specialized Gauss-to-Gauss transformations. In particular, the coverings $H_{32}-H_{35}$, H_{43} , H_{47} appear in the classical hypergeometric transformations listed by Goursat [8]. The coverings H_1 , H_2 , H_7 , H_8 , H_{11} , H_{18} , H_{42} appear in the hypergeometric transformations from $E(k, \ell, m)$ with k, ℓ, m positive integers satisfying $1/k + 1/\ell + 1/m < 1$. As determined in [26] (and [29, §9]), these pull-backs are unique up to Möbius transformations as well. The coverings H_{31} , H_{39} , H_{41} , H_{45} apply to hypergeometric transformations from $E(1/2, 1/2, \alpha)$ with infinite dihedral monodromy [31, §4]. The pulled-back equations have infinite cyclic or dihedral monodromy. They are, respectively,

$$E(1, 2\alpha, 2\alpha), \quad E(1/2, 1/2, 6\alpha), \quad E(1, 4\alpha, 4\alpha), \quad E(1/2, 1/2, 5\alpha).$$

The cyclic covering H_{48} gives the pull-back $E(1, \alpha, \alpha) \xleftarrow{4} E(1, 4\alpha, 4\alpha)$ of hypergeometric equations with infinite cyclic monodromy.

Non-unique Gauss-to-Gauss transformations appear when hypergeometric equations $E(k, \ell, m)$ are pulled-back, with k, ℓ, m positive integers satisfying $1/k + 1/\ell + 1/m \geq 1$. If the equality holds, these hypergeometric functions are integrals of holomorphic differentials on $j = 1728$ or $j = 0$ elliptic curves [29, §8]. Lemma 5.3 counts the coverings H_3 , H_{12} , H_{21} , H_{40} , H_{44} , H_{46} . It can be established (by identifying transformations of holomorphic differentials on the curves $y^2 = x^3 - 1$ and $x^3 + y^3 = 1$, $y^2 = x^6 + 1$) that the transformations from $E(1/2, 1/3, 1/6)$ to $E(1/3, 1/3, 1/3)$ or $E(2/3, 1/6, 1/6)$ are compositions of the pull-backs of Lemma 5.3 with quadratic transformations. This applies to the coverings H_{15} , H_{19} , H_{38} .

The hypergeometric equations $E(k, \ell, m)$ with $1/k + 1/\ell + 1/m > 1$ have finite monodromy groups. The hypergeometric solutions are thereby algebraic functions. These equations play a fundamental role in the classical theory of algebraic solutions of second order Fuchsian equations:

- $E(1, 1/k, 1/k)$, with the finite cyclic monodromy C_k .
- $E(1/2, 1/2, 1/k)$, with the dihedral projective monodromy D_k .
- $E(1/2, 1/3, 1/3)$, with the tetrahedral projective monodromy A_4 .
- $E(1/2, 1/3, 1/4)$, with the octahedral projective monodromy S_4 .
- $E(1/2, 1/3, 1/5)$, with the icosahedral projective monodromy A_5 .

By a celebrated theorem of Klein [14], all second order Fuchsian equations on \mathbb{P}^1 with a finite monodromy group are pull-backs of one of these standard hypergeometric equations, with the same projective monodromy group. These *Klein transformations* are known to be unique up to Möbius transformations [1]. However, pull-back transformations between hypergeometric equations with different projective monodromy need not to be unique. Litcanu [15, Theorem 2.1] noted non-uniqueness of the pull-backs from $E(1/2, 1/3, 1/4)$ to $E(1/2, 1/2, 1/2)$ and $E(1, 1/2, 1/2)$, of degree 6 and 12 respectively. The non-uniqueness is caused by pairs of different branching patterns though, e.g., $2+2+2=3+3=2+2+2$ and $2+2+1+1=3+3=4+2$. The example of $E(1/2, 1/2, 1/5) \xleftarrow{10} E(1/2, 1/2, 2)$ in [31, §5.4] shows that non-unique coverings with the same branching pattern easily occur for pull-backs to equations with apparent singularities. Besides, many compositions of

$$(5.2) \quad H_{37} : E(1/2, 1/3, 1/5) \xleftarrow{5} E(1/2, 1/3, 1/3)$$

with transformations from the tetrahedral equation are not unique either, because the properly normalized H_{37} is defined over $\mathbb{Q}(\sqrt{-15})$; see formula [29, (50)].

In Table 4, the coverings $H_9, H_{10}, H_{13}, H_{16}, H_{22}, H_{24}$ give Klein transformations of $E(1/2, 1/3, 1/5)$ to the following hypergeometric equations, respectively:

$$\begin{aligned} E(1/3, 1/5, 4/5), \quad E(1/3, 2/5, 3/5), \quad E(1/2, 1/5, 3/5), \\ E(2/3, 1/5, 2/5), \quad E(1/2, 1/3, 2/5), \quad E(1/3, 2/3, 1/5). \end{aligned}$$

This illustrates the Schwarz types VIII, XV, IX, X, XIV, XII, respectively. The other icosahedral Schwarz types are represented by $E(1/3, 1/3, 2/5)$, $E(1/5, 1/5, 4/5)$, $E(2/5, 2/5, 2/5)$, and the standard $E(1/2, 1/3, 1/5)$. Uniqueness of the coverings $H_{14}, H_{17}, H_{23}, H_{29}, H_{30}$ is established by noting these Klein transformations:

$$\begin{aligned} E(1/2, 1/3, 1/3) &\xleftarrow{9} E(1/2, 2/3, 4/3), \quad E(1/2, 1/2, 1/3) \xleftarrow{8} E(3/2, 3/2, 2/3), \\ E(1/2, 1/3, 1/4) &\xleftarrow{7} E(1/2, 1/3, 3/4), \quad E(1/2, 1/3, 1/4) \xleftarrow{5} E(1/2, 2/3, 1/4), \\ &E(1/2, 1/3, 1/3) \xleftarrow{5} E(1/2, 2/3, 2/3). \end{aligned}$$

These considerations of reduction to hypergeometric transformations do not immediately establish uniqueness of 10 coverings in Table 4. Those coverings induce rather attractive transformations between hypergeometric equations with different finite monodromy. In particular, H_6, H_{28} pull-back $E(1/2, 1/3, 1/3)$ to $E(1, 1, 1)$ and $E(1, 1/2, 1/2)$; then $H_5, H_{20}, H_{25}, H_{27}, H_{36}$ transform $E(1/2, 1/3, 1/4)$ to $E(1, 1/2, 1/2)$, $E(1, 1/3, 1/3)$, $E(1/2, 1/3, 2/3)$, $E(1/2, 1/2, 1/2)$, $E(1/2, 1/2, 1/3)$, respectively; and finally, H_4, H_{26}, H_{37} pull-back $E(1/2, 1/3, 1/5)$ to $E(1, 1/5, 1/5)$, $E(1/2, 1/2, 1/5)$ and (5.2). Many of the coverings pull-back $E(1/2, 1/3, 1/2)$ or other dihedral hypergeometric equations to hypergeometric equations with simpler dihedral or cyclic monodromy.

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REFERENCES

- [1] F. Baldassarri and B. Dwork. On second order linear differential equations with algebraic solutions. *Amer. J. Math.*, 101(1):42–76, 1979.
- [2] A. Beauville. Les familles stables de courbes elliptiques sur \mathbf{P}^1 admettant quatre fibres singulières. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(19):657–660, 1982.
- [3] F. Beukers. Gauss’ hypergeometric functions. In R.-P. Holzapfel, A. M. Uludag, and M. Yoshida, editors, *Arithmetic and Geometry Around Hypergeometric Functions*, Progress in Mathematics Series No 260, pages 23–42. Birkhäuser, Boston/Basel, 2007.
- [4] F. Beukers and H. Montanus. Explicit calculation of elliptic fibrations of $K3$ -surfaces and their Belyi-maps. In J. McKee and C. Smyth, editors, *Number Theory and Polynomials*, LMS Lecture Note Series No 352, pages 33–51. Cambridge Univ. Press, 2008. Supplementary data available at <http://www.math.uu.nl/people/beukers/mirandapersson/Dessins.html>.
- [5] S. A. Broughton, D. M. Haney, L. T. McKeough, and B. S. Mayfield. Divisible tilings in the hyperbolic plane. *New York J. Math.*, 6:237–283, 2000.

- [6] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, editors. *Higher Transcendental Functions*. McGraw-Hill, New York, 1953–55. Also known as *The Bateman Manuscript Project*.
- [7] A. A. Felikson. Coxeter decompositions of hyperbolic polygons. *European J. Combin.*, 19(7):801–817, 1998.
- [8] É. Goursat. Sur l'équation différentielle linéaire qui admet pour intégrale la série hypergéométrique. *Ann. Sci. École Normale Sup. (2)*, 10:S3–S142, 1881.
- [9] R. Hartshorne. *Algebraic Geometry*. Number 52 in Graduate Texts in Mathematics. Springer-Verlag, New York/Berlin, 1977.
- [10] S. Herfurtnier. Elliptic surfaces with four singular fibres. *Math. Ann.*, 291(2):319–342, 1991.
- [11] J. Hodgkinson. An application of conformal representation to certain hypergeometric series. *Proc. London Math. Soc. (2)*, 17:17–24, 1918.
- [12] J. Hodgkinson. A detail in conformal representation. *Proc. London Math. Soc. (2)*, 18:268–274, 1920.
- [13] A. V. Kitaev, Grothendieck's Dessins d'Enfants, Their Deformations and Algebraic Solutions of the Sixth Painlevé and Gauss Hypergeometric Equations, *Algebra i Analiz* **17**, no. 1 (2005), 224–273 (<http://xxx.lanl.gov>, nlin.SI 0309078, 1–35, 2003).
- [14] F. Klein, Über lineare Differentialgleichungen I, *Math. Annalen* **11** (1877), 115–118.
- [15] R. Litcanu, Lamé operators with finite monodromy – a combinatorial approach, *J. Differential Equations*, 207 (2004): 93–116.
- [16] R. S. Maier. On reducing the Heun equation to the hypergeometric equation. *J. Differential Equations*, 213 (2005): 171–203.
- [17] R. S. Maier. P-symbols, Heun identities, and ${}_3F_2$ identities. In D. Dominici and R. S. Maier, editors, *Special Functions and Orthogonal Polynomials*, number 471 in Contemporary Mathematics, pages 139–159. American Mathematical Society (AMS), Providence, RI, 2008.
- [18] R. Miranda and U. Persson. Configurations of I_n fibers on elliptic K3 surfaces. *Math. Z.*, 201:339–361, 1989.
- [19] H. Movasati and S. Reiter. Heun equations coming from geometry. Preprint, available as arXiv:0902.0760 [math.AG], 2009.
- [20] E. G. C. Poole. *Introduction to the Theory of Linear Differential Equations*. Oxford Univ. Press, Oxford, UK, 1936.
- [21] A. Ronveaux, editor. *Heun's Differential Equations*. Oxford Univ. Press, Oxford, UK, 1995. With contributions by F. M. Arscott, S. Yu. Slavyanov, D. Schmidt, G. Wolf, P. Maroni and A. Duval.
- [22] L. Schneps. Dessins d'enfants on the Riemann sphere. In L. Schneps, editor, *The Grothendieck Theory of Dessins d'Enfants*, number 200 in London Mathematical Society Lecture Note Series, pages 47–77. Cambridge Univ. Press, Cambridge, UK, 1994.
- [23] H.A. Schwarz, Ueber diejenigen Falle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Funktion ihres vierten Elements darstellt, *Journ. für die reine und angewandte Math.* **75** (1872), 292–335.
- [24] G. Shabat. On a class of families of Belyi functions. In D. Krob, A. A. Mikhalev, and A. V. Mikhalev, editors, *Formal Power Series and Algebraic Combinatorics (Moscow, 2000)*, pages 575–580. Springer-Verlag, New York/Berlin, 2000.
- [25] T. Shioda, Elliptic surfaces and Davenport-Stothers triples. *Comment. Math. Univ. St. Pauli* **54** (2005), no. 1, 49–68.
- [26] R. Vidunas. Transformations of some Gauss hypergeometric functions. *J. Comput. Appl. Math.*, 178(1–2):473–487, 2005.
- [27] R. Vidunas, Degenerate Gauss hypergeometric functions, *Kyushu J. Math.* **61** (2007), p. 109–135.
- [28] R. Vidunas. Dihedral Gauss hypergeometric functions. *Kyushu J. Math.* **65** (2011), 141–167
- [29] R. Vidunas. Algebraic transformations of Gauss hypergeometric functions. *Funkcial. Ekvac.*, 52:139–180, 2009.
- [30] R. Vidunas, G. Filipuk. Parametric transformations between the Heun and Gauss hypergeometric functions. Submitted to *Funkcialaj Ekvacioj*, 2012. Available at <http://arxiv.org/abs/0910.3087v2>
- [31] R. Vidunas. Transformations and invariants for dihedral Gauss hypergeometric functions. *Kyushu J. Math.* **66** (2012), 143–170
- [32] R. Vidunas, *Heun equations with cyclic or dihedral monodromy*, under preparation.

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